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Motivic decomposition of abelian schemes and the Fourier transform. (English)

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For a field k, let S be a smooth quasi-projective (connected) scheme over k. Write $\mathcal{V}(S)$ for the category of smooth S-schemes $\lambda : X \to S$. The usual construction of the category of (Chow) motives over a field extends to the situation of S-schemes. One obtains categories $\mathcal{M}^0_+(S)$ and $\mathcal{M}(S)$ of *effective relative Chow motives*. These are constructed as the Karoubian (pseudo-abelian) envelopes of $\mathcal{V}(S)$ with morphisms given by the graded (resp. ungraded) correspondences $CH^{dim(X/S)}(X\times_S Y,\mathbb{Q}) = CH^{dim(X/S)}(X\times_S Y)\otimes\mathbb{Q}$ (resp. $CH(X \times_S Y,\mathbb{Q}) = CH(X \times_S Y) \otimes \mathbb{Q}$). For $\lambda : X \to S$ in $\mathcal{V}(S)$ one writes $R(X/S)$ for (X,id) in $\mathcal{M}(S)$, and if π_i is a projector, one writes $R^i(X/S)$ for the motive (X,π_i). In particular, one has a Lefschetz motive $L_S = R^2(\mathbb{P}^1_S/S)$ such that for any X with connected fibres of dimension d and canonical projector $\pi_{2d} = X \times_S e(S)$, $e : S \to X$ a section, there is an isomorphism $R^{2d}(X/S) \rightsquigarrow L_S^{\otimes d}$. Localization of $\mathcal{M}^0_+(S)$ with respect to $M \mapsto M\otimes L_S$ gives the category $\mathcal{M}^0(S)$ of *Chow motives* with respect to graded correspondences. Tensoring with $L_S^{\otimes -m} = (L_S^{-1})^{\otimes m}$ defines the twists $M(m)$, $m \in \mathbb{Z}$. Bloch's Chow groups $CH^\bullet(-,j)$ and motivic cohomology $H^\bullet_{\mathcal{M}}(-,\mathbb{Q}(j)) = CH^j(-,2j-\bullet)\otimes\mathbb{Q}$ factor over $\mathcal{M}^0(S)$ and ones has e.g. $CH^i(M(m),j) = CH^{i+m}(M,j)$. $\mathcal{M}^0(S)$ and $\mathcal{M}(S)$ behave well under base change, and the functor $R_\ell : \mathcal{V}(S) \to D^b(S,\mathbb{Q}_\ell)$, $(\lambda : X \to S) \mapsto R\lambda_*\mathbb{Q}_\ell$, extends to a $\mathbb{Q}$-linear functor $R_\ell : \mathcal{M}^0(S) \to D^b(S,\mathbb{Q}_\ell)$ which commutes with twists and tensor products and is compatible with base extension. Here $D^b(S,\mathbb{Q}_\ell)$ denotes the bounded derived category of $\mathbb{Q}_\ell$-sheaves on S.

A general problem in the theory of motives is to find a decompositon of the form $R(X/S) = \oplus_i R^i(X/S)$ for suitable projectors π_i and to describe the components $R_i(X/S)$, e.g. what are their realizations? In the underlying paper a canonical functorial decomposition of $R(A/S)$, where A is an abelian scheme over S, is established by means of the *Fourier transform*

$F : \mathcal{M}(S) \to \mathcal{M}(S)$, $F = F_A : R(A/S) \mapsto R(\hat{A}/S)$, where $\hat{A}/S$ is the dual abelian scheme. F is defined as the correspondence $F = F_A \in CH(A \times_S \hat{A},\mathbb{Q})$ given by the formula

$$F = F_A = ch(L) = \exp(c_1(\mathcal{L})) = 1 + \frac{c_1(\mathcal{L})}{1!} + \frac{c_1(\mathcal{L})^2}{2!} + \cdots ,$$

where $\mathcal{L}$ is a Poincaré line bundle on $A\times_S\hat{A}$ with class $L \in Pic(A\times_S\hat{A})$ rigidified along the zero sections, and where $c_1(\mathcal{L}) \in CH^1(A \times_S \hat{A})$ is the associated (Chern) divisor class. For the Fourier transform $\hat{F} : \hat{A} \to \hat{\hat{A}} = A$ one has $\hat{F} = {}^tF$, and with the map $\sigma : A \to A, a \mapsto -a$, one has $\hat{F} \circ F = (-1)^g[\Gamma_\sigma]$, where g is the fibre dimension of A/S and $[\Gamma_\sigma]$ is the class of the graph of σ. Thus F is an automorphism of $\mathcal{M}(S)$ with inverse $F^{-1} = (-1)^g[\Gamma_\sigma] \circ \hat{F}$. Also, for an isogeny $f : A \to B$ between abelian schemes A/S and B/S, one has commutativity in $\mathcal{M}(S)$:

(i) $F_A \circ f^* = \hat{f}_* \circ F_B$ and

(ii) $F_B \circ f_* = \hat{f}^* \circ F_A$.

F defines a homomorphism $F_{CH} : CH(A,\mathbb{Q}) \to CH(\hat{A},\mathbb{Q})$ by $F_{CH}(\xi) = p_{2*}(p_1^*(\xi)\cdot F)$ and (i) and (ii) carry over to corresponding properties of F_{CH}. The F_{CH} play a main role in the proof of the final result of the paper:

Theorem 1. Let $\lambda : A \to S$ be an abelian scheme of fibre dimension g and let $n : A \to A$ be multiplication by n. Then the diagonal $\Delta = \Delta(A/S)$ has a unique decomposition $\Delta = \sum_{i=0}^{2g}\pi_i$ in $CH^g(A \times_S A,\mathbb{Q})$, where the π_i are pairwise orthogonal idempotents, such that $(id_A \times n)^*\pi_i = n^i\pi_i$ for all $n \in \mathbb{Z}$. Moreover, $[{}^t\Gamma_n] \circ \pi_i = \pi_i \circ [{}^t\Gamma_n] = n^i\pi_i$.

Corollary 2. (i) The motive $R(A/S)$ decomposes as $(*)$ $R(A/S) = \bigoplus_{i=0}^{2g} R^i(A/S)$, where $R^i(A/S) = (R(A/S),\pi_i)$ are the relative Chow motives in $\mathcal{M}^0(S)$ determined bz π_i. n^* acts by multiplication with n_i on $R^i(A/S)$.

(ii) For the ℓ-adic realization of $R(A/S)$ one obtains $R^i_\ell(R^j(A/S)) = 0$, for $i \neq j$, $R^i_\ell(R^j(A/S)) = R^j\lambda^*\mathbb{Q}_\ell$, for $i = j$.

(iii) The decomposition $(*)$ induces a canonical splitting in $D^b(S, \mathbb{Q}_\ell)$:

$$R\lambda_* \mathbb{Q}_\ell \cong \bigoplus_{i=0}^{2g} R^i \lambda_* \mathbb{Q}_\ell[-i].$$

The paper closes with a remark on the motive H_χ attached to a Hecke character χ of a number field K with values in a number field E. In this case, there exists a Chow motive M'_χ over $S = \mathrm{Spec}(K)$ with coefficients in an extension E' of E such that for the absolute Hodge realization ${M'}_\chi^{a.H.}$ of M'_χ one has ${M'}_\chi^{a.H.} = H_\chi \times_E E'$.

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MSC:

14A20 Generalizations (algebraic spaces, stacks)
14K05 Algebraic theory of abelian varieties
14C35 Applications of methods of algebraic K-theory in algebraic geometry
14K02 Isogeny
14C05 Parametrization (Chow and Hilbert schemes)

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