Soulé, C.; **Abramovich, Dan**; **Burnol, J.-F.**; **Kramer, Jürg**
Lectures on Arakelov geometry. (English) Zbl 0812.14015
Cambridge Studies in Advanced Mathematics. 33. Cambridge: Cambridge University Press. 177 p. (1992).

Let X be an arithmetic variety and $\overline{E}$ an hermitian vector bundle on X. We shall attach to $\overline{E}$ characteristic classes with values in arithmetic Chow groups. More specifically, an arithmetic cycle is a pair (Z,g) consisting of an algebraic cycle on X, i.e. a finite sum $\sum_\alpha n_\alpha Z_\alpha$, $n_\alpha \in \mathbb{Z}$, where Z_α is a closed irreducible subscheme of X, of fixed codimension p, say, and a Green current g for Z. By this we mean that g is a real current on X_∞ which satisfies $F_\infty^*(g) = (-1)^{p-1}g$ and $dd^c g + \delta_Z = \omega$, where ω is (the current attached to) a smooth form on X_∞, and δ_Z is the current given by integration on Z_∞: $\delta_Z(\eta) = \sum_\alpha n_\alpha \int_{Z_\alpha(\mathbb{C})} \eta$, for any smooth form η of appropriate degree. The arithmetic Chow group $\widehat{CH}^p(X)$ is the abelian group of arithmetic cycles, modulo the subgroup generated by pairs $(0, \partial u + \overline{\partial} v)$ and $(\operatorname{div} f, -\log|f|^2)$, where u and v are arbitrary currents of the appropriate degree and div f is the divisor of a nonzero rational function f on some irreducible closed subscheme of codimension $p-1$ in X.

In chapter III we study the groups $\widehat{CH}^p(X)$, showing that they have functoriality properties and a graded product structure, at least after tensoring them by $\mathbb{Q}$. To prove these facts is rather difficult, for two reasons. First, the intersection theory on a general regular scheme such as X cannot be defined in the usual way, since no moving lemma is available. We remedy this in chapter I by using algebraic K-theory and Adams operations. – A second difficulty is that, given two arithmetic cycles (Z,g) and (Z',g'), we need a Green current for their intersection. The formula $g'' = \omega g' + g\delta_{Z'}$ is formally satisfactory, but involves a product of currents $g\delta_{Z'}$. To make sense of it in general we need to show that we can take for g a smooth form on $X_\infty - Z_\infty$, of logarithmic type along Z_∞. This is done in chapter II.

After having set up arithmetic intersection theory, we define in chapter IV characteristic classes for hermitian vector bundles $\overline{E}$ on X. – Our next construction is some direct image map for hermitian vector bundles. Let $f: X \to Y$ be a proper flat map between arithmetic varieties, smooth on the generic fiber $X_\mathbb{Q}$. According to *F. Knudsen* and *D. Mumford* [Math. Scand. 39, 19-55 (1976; Zbl 0343.14008)], there is a canonical line bundle $\lambda(E)$ on Y whose fiber at every point $y \in Y$ is the determinant of the cohomology of $X_y = f^{-1}(y)$ with coefficients in E: $\lambda(E)_y = \bigotimes_{q\geq 0} \bigwedge^{\max}(H^q(X_y,E))^{(-1)^q}$. To get a metric on $\lambda(E)$ let us fix a Kähler metric on X_∞, hence on each fiber X_y, $y \in Y_\infty$. According to *D. Quillen* [Funct. Anal. Appl. 19, 31-34 (1985); translation from Funkts. Anal. Prilozh. 19, No. 1, 37-41 (1985; Zbl 0603.32016)] we may then get a smooth metric h_Q on $\lambda(E)_\infty$ by multiplying its L^2-metric by the exponential of the Ray- Singer analytic torsion: $h_Q = h_{L^2} \cdot \exp(T(E))$, with $T(E) = \sum_{q\geq 0}(-1)^{q+1} q\zeta_q'(0)$. Here $\zeta_q'(0)$ is the derivative at the origin of the zeta function $\zeta_q(s)$, $s \in \mathbb{C}$. – In chapter V we study $\zeta_q(s)$ and the Quillen metric. Following *J. M. Bismut*, *H. Gillet* and *C. Soulé* [Commun. Math. Phys. 115, No. 1, 49-78; 79-126; No. 2, 301-351 (1988; Zbl 0651.32017)] we show that h_Q is smooth and compute the curvature on Y_∞ of the hermitian line bundle $\lambda(E)_Q = (\lambda(E), h_Q)$. It is given by a Riemann-Roch-Grothendieck formula at the level of forms. – When combining the above results with the Riemann-Roch-Grothendieck theorem for algebraic Chow groups, we get in chapter VI a Riemann-Roch-Grothendieck theorem for arithmetic Chow groups.

MSC:

14G40	Arithmetic varieties and schemes; Arakelov theory; heights
14C05	Parametrization (Chow and Hilbert schemes)
14C40	Riemann-Roch theorems

Keywords:

arithmetic variety; arithmetic Chow groups; direct image map for hermitian vector bundles; zeta function; Quillen metric; Riemann-Roch- Grothendieck theorem; moving lemma