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Lectures on vanishing theorems. Notes, grew out of the DMV-seminar on algebraic geometry, held at Reisensburg, October 13-19, 1991. (English) Zbl 0779.14003

DMV Seminar. 20. Basel: Birkhäuser Verlag. 164 p. (1992).

The cohomology groups of sheaves over an algebraic variety embody a basic algebraic tool for the geometric study of the variety itself. In particular, the vanishing of certain cohomology groups for distinguished sheaves allows to conclude, from the fundamental general cohomology theorems and sequences, more specific information about the existence, quantity and quality of characterizing objects associated with the variety under investigation, about its classifying numerical invariants and about its deformation behaviour. In regard to this special theoretical and practical significance, general vanishing theorems for sheaf cohomology are of crucial importance and, quite obviously, of far- reaching utility.

One of the very first vanishing theorems, in this direction, is the famous Kodaira vanishing theorem which states that the inverse of an ample line bundle over a projective complex manifold has zero cohomology below the dimension of the manifold. Kodaira proved his theorem in 1953, and little later, in 1954, Akizuki and Nakano obtained a generalization of it. The proofs for these vanishing theorems were based on methods of (Hermitean) differential geometry, and – although Serre's GAGA-theorems established the validity of the Kodaira-Akizuki-Nakano vanishing theorem also for projective manifolds over an algebraically closed groundfield of characteristic zero – for more than 25 years no direct algebraic approach to them was found. Even worse, in 1978 M. Raynaud constructed a counterexample in characteristic $p > 0$, thus showing that the Kodaira- Akizuki-Nakano vanishing theorem, under its original assumptions, does not generalize to arbitrary projective manifolds, and that a purely algebraic approach needed some additional assumptions. In the meantime, several generalizations of the (complex-analytic) Kodaira-Akizuki-Nakano theorem had been discovered, but the crucial break-through (towards an algebraization) was achieved in 1986-1987, when the relations between the Hodge and the de Rham spectral sequences, on the one hand, and the vanishing theorems, on the other hand, became apparent, due to the work of the authors [Invent. Math. 86, 161-194 (1986; Zbl 0603.32006)]and *P. Deligne* and *L. Illusie* [Invent. Math. 89, 247-270 (1987; Zbl 0632.14017)]. In the latter work, Deligne and Illusie proved the degeneration of Hodge spectral sequence to the de Rham spectral sequence for projective manifolds over fields of characteristic $p > 0$ and liftable to the ring of second Witt vectors. Reduction modulo p gave the same result in characteristic zero, and this, ultimately, provided the algebraic approach that could replace the differential-geometric methods used before.

The combination of the authors' methods and those by Deligne and Illusie (loc. cit.) gives another algebraic approach to vanishing theorems, which implies, in a systematic and unified way, many of the known versions, special cases and generalizations; for example those ones obtained by Grauert-Riemenschneider, Ramanujam, Miyaoka, Kawamata-Viehweg, Bogomolov, Sommese, Kollár, and others. – It is one of the aims of the present book to work out this combined approach of the authors and Deligne- Illusie in full detail, as algebraic as possible, and with all its relations and applications to the whole theory of vanishing theorems developed so far. More than that, as the book is an extended elaboration of lecture notes grown out of a crash course for graduate students and younger researchers in algebraic geometry, it also provides a comprehensive, lucid, systematic and rigorous introduction to the whole subject and its methodical framework. The content consists of thirteen sections (lectures) and one appendix. After a motivating discussion of Kodaira's vanishing theorem from various viewpoints, including and outlining the possible proofs of it, the basic framework such as logarithmic de Rham complexes, integral parts of $\mathbb{Q}$-divisors and coverings, and the formal set-up for vanishing theorems via degenerations of spectral sequences is thoroughly developed. In addition, the fundamental algebraic vanishing theorems arising from this approach are derived. Then the geometric interpretation of them, i.e., the discussion of the various known generalizations of the Kodaira-Akizuki-Nakano vanishing theorem within this context, is carried out. After some more applications of the obtained vanishing theorems to specific geometric situations, yielding further (partially known) results in this area, the characteristic p methods and lifting properties are explained, mainly in order to provide the prerequisites for the proof of the crucial degeneration theorem of Deligne-Illusie (loc. cit.). The proof thereof is presented, in necessary completeness, in the most spacious section of the book.

In the sequel, the elegant algebraic proof of the Kodaira-Akizuki-Nakano vanishing theorem, based on this approach, is deduced, and the text concludes with a discussion of the deformation theory for cohomology groups. This touches the very recent progress achieved in the theory of vanishing theorems, chiefly the generic vanishing results of *M. Green* and *R. Lazarsfeld* [Invent. Math. 90, 389-407 (1987; Zbl 0659.14007) and J. Am. Math. Soc. 4, No. 1, 87-103 (1991; Zbl 0735.14004)]; cf. also *H. Dunio* "Über generische Verschwindungssätze", Diplomarbeit (Univ. Essen 1991). The appendix compiles the basic formal properties of the cohomology of complexes that are used throughout the text.

To sum up, the book under review is a perfect introduction to the theory of algebraic vanishing theorems, which leads the reader – along a masterly written text — to the frontiers of research in this area, through all the powerful (but fairly intricate) avanced framework. The book has been written by two of the very experts and main contributors in this field of research, and their many hints, remarks and references towards related results and methods make the book also a highly valuable source for actively working specialists. The text is basically self- contained, provided the reader has a firm knowledge of the fundamentals of modern algebraic geometry.

Reviewer: W.Kleinert (Berlin)

MSC:

14F17	Vanishing theorems in algebraic geometry
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
14F40	de Rham cohomology and algebraic geometry
14D15	Formal methods and deformations in algebraic geometry
14F30	p-adic cohomology, crystalline cohomology
14F10	Differentials and other special sheaves; D-modules; Bernstein-Sato ideals and polynomials

Keywords:

Reisensburg (Germany); DMV Seminar; Hodge sequences; characteristic p; vanishing theorems; de Rham spectral sequences; logarithmic de Rham complexes