Masser, David; **Wüstholz, Gisbert**

Periods and minimal abelian subvarieties. (English) Zbl 0796.11023

Ann. Math. (2) 137, No. 2, 407-458 (1993).

In this text, the authors prove a very important albeit rather technical estimate on abelian subvarieties of a given abelian variety. More precisely, the situation is the following: let A be a polarized abelian variety defined over a number field k, viewed as a subfield of the field of complex numbers $\mathbb{C}$, and consider an element ω of the period lattice of the tangent space at the origin of $A(\mathbb{C})$. Then, the degree of the smallest abelian subvariety $\mathbb{G}_\omega$ (for the projective embedding of $\mathbb{G}_\omega$ induced by the polarization of A composed with the inclusion map) of A containing ω in its tangent space is bounded by ch^κ, where h is the Faltings height of A, the constant κ is effectively computable in terms of the dimension n of A, and c is of the form $c_1(n)(d\delta)^{\kappa_1}$, where d is the degree of k over $\mathbb{Q}$ and δ is the degree of the polarization on A, and κ_1 (but not c_1) is effectively computable in terms of n.

To illustrate the usefulness of this result, let us describe the following application: let A/k be an abelian variety defined over k, of Faltings height bounded by h, then if A^*/k is another abelian variety defined over k, isogenous to A, there is an isogeny from A to A^* of degree bounded polynomially in terms of h. This result also due to the authors can be found in Ann. Math., II. Ser. 137, 459-472 (1993). This latter result can be applied to get an almost effective version of a theorem of *J.-P. Serre* [Invent. Math. 15, 259-331 (1972; Zbl 0235.14012)] on the Galois group of division fields of elliptic curves [the authors, Bull. Lond. Math. Soc. 25, 247-254 (1993)].

The proof of the main theorem is as follows. One first remarks that studying the group $\mathbb{G}_\omega$ is equivalent to studying linear dependence relations over $\overline{\mathbb{Q}}$ of the coordinates of ω (over a suitable basis of the tangent space), which in turn is an "effective" Baker or, more precisely "analytic subgroup theorem" (Wüstholz). This is for the transcendence part of the proof (i.e. only the last sections (9, 10) of the paper).

The rest of the paper is devoted to the necessary auxiliary lemmas: after describing the Siegel moduli space, the complex uniformization, the authors work with theta functions. After giving effective versions of the addition formula (derived from Riemann relations), they show that for a suitable basis (introduced by Shimura, and later studied by the reviewer [Compos. Math. 78, 121-160 (1991; Zbl 0741.14025)]) of the tangent space, the differential equations satisfied by abelian functions are algebraic, of bounded height (in terms of the height of A). Their proof is different from the one provided by the one in the latter reference, but also based on the modular properties of these differentials.

They then generalize various estimates first obtained by *D. Masser* [on the norm of the period matrix of A etc. ..., see Lect. Notes Math. 1290, 109-148 (1987; Zbl 0639.14025)], and an analytic growth estimate of theta functions [the reviewer (loc. cit.)]. They then prove a crucial lemma (section 8) which shows that the Grassmann height of the tangent space of an abelian subvariety is bounded in terms of the height of A and (the log of) the degree of the subvariety. This in turn uses another estimate (the "denominators lemma" of section 5) which shows that essentially the "Shimura" basis of the tangent space of A is an integral basis. Once these estimates are available the routine of the transcendence machine almost gives the required result. Some minor degree computations are enough to conclude. This paper thus proves a result which has already led (and doubtless in the future will lead) to interesting developments and applications.

Reviewer: S.David (Paris)

MSC:

11G10 Abelian varieties of dimension > 1
14K15 Arithmetic ground fields for abelian varieties
11J89 Transcendence theory of elliptic and abelian functions

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analytic subgroup theorem; abelian variety; abelian subvariety; Faltings height

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