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Relative deformations of morphisms and applications to fibre spaces. (English) Zbl 0813.14001

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The author introduces a notion of relative deformation for a morphism $f:Y\to X$ between smooth algebraic varieties, and he applies it to study the case where the fibers of f are Fano manifolds. – Let k be a field, $f:Y\to X$ and $\pi:Z\to X$ k-morphisms between smooth k-varieties, and (S,o) a connected punctured k-scheme. An S- morphism $\widetilde{f}:Y\times_k S\to X\times_k S$, or a family $\widetilde{f}=\{f_s:Y\to X\}_{s\in S}$, is a "relative deformation" of f over Z with base subscheme $B\subset Y$, if the following three conditions are satisfied:

$$\widetilde{f}_{|Y\times\{0\}}=f,\quad \widetilde{f}_{|B\times S}=(f_{|B},\mathrm{pr}_S)\quad\text{and}\quad(\pi,\mathrm{pr}_S)\circ\widetilde{f}=(\pi\circ f,\mathrm{pr}_S).$$

We denote by $\mathrm{Deform}_Z(S,o;f,B)$ the set of relative deformations of f over Z with base B, and we define a contravariant functor from the category of punctured k-schemes to the category of sets: $\mathrm{Deform}_Z(*;f,B):(S,o)\mapsto\mathrm{Deform}_Z(S,o;f,B)$. We denote by $\mathrm{Sing}(\pi)$ the singular locus of π, $f^\# T_{X/Z}$ the kernel of $f^*d\pi: f^*T_X\to f^*\pi^*T_Z$.

Theorem 1. Assume that X and Y are both projective and that π is surjective.

(1) The functor $\mathrm{Deform}_Z(*;f,B)$ is representable by a quasi-projective k-scheme $D_Z(f,B)$, the universal Z-deformation of f with base B.

(2) If $\dim_k(Y)=1$ and $f(Y)$ is not contained in $\mathrm{Sing}(\pi)$, then the tangent space of $D_Z(f,B)$ at $[f]=0$ is $H^0(Y,\mathcal{I}_B f^\# T_{X/Z})$ and the obstruction lies in $H^1(Y,\mathcal{I}_B f^\# T_{X/Z})$.

(3) When k is the fraction field of a ring finitely generated over $\mathbb{Z}$ the statements (1) and (2) hold on almost every reduction of positive characteristic.

To prove the first part of the theorem, we use the theory of the Hilbert schemes and for the second part we study the sheaves of differential operators on X and Z, and we look at infinitesimal deformations. Then we can use standard arguments on deformation. We have the following applications:

Theorem 2. Let X and Z be smooth projective k-varieties; let $\pi:X\to Z$ be a surjective projective morphism with at least one smooth fiber. Then the relative anti-canonical sheaf $\omega^{-1}_{X/Z}$ cannot be ample, unless Z is a single point.

Theorem 3. Let $\pi:X\to Z$ be a morphism as above with X a Fano variety; let H be an ample divisor on Z, $\alpha>0$ such that $-K_X-\alpha\pi^*H$ is nef and let C be an irreducible curve on Z. If $(C,-K_Z-\alpha H)<0$, then C is contained in the discriminant locus of π.

If we assume that a morphism $f:Y\to X$, where Y is a smooth projective curve, has a nontrivial deformation over Z, by the theorem 1 we can show there exists a morphism $f':Y\to X$ with $\deg f'(Y)<\deg f(Y)$ and $\pi\circ f'=\pi\circ f$. – Then to prove theorems 2 and 3 we use Mori's arguments: we make reduction modulo p and we replace the map f by the composite with a suitable geometric Frobenius.

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MSC:

14B12 Local deformation theory, Artin approximation, etc.
14F10 Differentials and other special sheaves; D-modules; Bernstein-Sato ideals and polynomials
14J45 Fano varieties
14E05 Rational and birational maps
14C05 Parametrization (Chow and Hilbert schemes)

Keywords:

relative deformation for a morphism; Fano manifold; Hilbert schemes; sheaves of differential operators; infinitesimal deformations; Fano variety