Brauer, Richard

Author’s introduction: Consider a system of algebraic equations

\[ \begin{align*}
    f_1(x_1, x_2, \ldots, x_n) &= 0, \\
    f_2(x_1, x_2, \ldots, x_n) &= 0, \\
    \vdots \\
    f_h(x_1, x_2, \ldots, x_n) &= 0,
\end{align*} \tag{1} \]

where \( f_i \) is a homogeneous polynomial of degree \( r_i \) with coefficients belonging to a given field \( K \). We interpret homogeneous coordinates in an \((n - 1)\)-dimensional projective space. When \( n > h \), the system (1) has non-trivial solutions \((x_1, x_2, \ldots, x_n)\) in an algebraically closed extension field of \( K \), but there may not exist any such solutions in \( K \) itself. It is, in general, extremely difficult to decide whether adjunction of irrationalities of a certain type to \( K \) is sufficient to guarantee the existence of non-trivial solutions of (1) in the extended field. However, the situation is much simpler, when \( n \) is very large, in the sense that \( n \) lies above a certain expression depending on the number of equations \( h \) and the degrees \( r_1, r_2, \ldots, r_h \).

We show:

Theorem A. For any system of \( h \) positive degrees \( r_1, r_2, \ldots, r_h \), there exists an integer \( \Phi(r_1, r_2, \ldots, r_h) \) such that for \( n \geq \Phi(r_1, r_2, \ldots, r_h) \) the system (1) has a non-trivial solution in a soluble extension field \( K_1 \) of \( K \). The field \( K_1 \) may be chosen such that its degree \( N_1 \) over \( K \) lies below a value depending on \( r_1, r_2, \ldots, r_h \) alone and that any prime factor of \( N_1 \) is at most equal to \( \max(r_1, r_2, \ldots, r_h) \).

This Theorem A is evidently contained in the following.

Theorem B. For any system of positive integers \( r_1, r_2, \ldots, r_h \) and any integer \( m \geq 0 \), there exists an integer \( \Phi(r_1, r_2, \ldots, r_h; m) \) with the following property. For \( n \geq \Phi(r_1, r_2, \ldots, r_h; m) \), there exists a soluble extension field \( K_2 \) of \( K \) such that all points \( x_1, x_2, \ldots, x_n \) of an \( m \)-dimensional linear manifold \( L \), defined in \( K_2 \), satisfy the equations (1). Here \( K_2 \) may be chosen so that its degree \( N_2 \) over \( K \) lies below a bound depending on \( r_1, r_2, \ldots, r_h \) and \( m \) and that no prime factor of \( N_2 \) exceeds \( \max(r_1, r_2, \ldots, r_h) \).

At the same time, we prove the theorem:

Theorem C. Assume that the field \( K \) has the following property,

(*) For every integer \( r > 0 \), there exists an integer \( \Psi(r) \) such that for \( n \geq \Psi(r) \) every equation

\[ a_1x_1^r + a_2x_2^r + \cdots + a_nx_n^r = 0 \tag{2} \]

with coefficients \( a_i \in K \) has a non-trivial solution in \( K \).

Then, for every system of positive degrees \( r_1, r_2, \ldots, r_h \), and every integer \( m \geq 0 \), there exists an expression \( \Omega(r_1, r_2, \ldots, r_h; m) \) with the following property: For \( n \geq \Omega(r_1, r_2, \ldots, r_h; m) \), there exists an \( m \)-dimensional linear manifold \( M \), defined in \( K \), whose points satisfy the equations (1).

We prove Theorem C in §2. The changes necessary in order to obtain Theorem B are obvious. In §3, some applications are given. One of them is concerned with Hilbert’s resolvent problem. We prove here a recent conjecture of B. Segre [Ann. Math. (2) 46, 287–301 (1945; Zbl 0061.01807)].

MSC:

11D72 Diophantine equations in many variables

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