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Rational curves on algebraic varieties. (English) Zbl 0877.14012

Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 32. Berlin: Springer-Verlag. viii, 320 p. (1995).

Recent progress in the structure theory of higher-dimensional varieties has developed into different directions. One of them concentrates on the study of the geometry of curves on varieties: Based on classical geometric ideas, the book under review investigates rational curves on varieties of arbitrary dimension. Main techniques are the use of Hilbert and Chow schemes. This is worked out in chapter I in the necessary generality, taking into account especially the case of positive characteristic where the Chow-functor does not behave as well as in characteristic 0. Also, there are hints and references to the cases of algebraic and analytic spaces, respectively, as well as an appendix with some background from commutative algebra.

Chapter II applies the preceding techniques to study curves on varieties and morphisms from curves to varieties: rational curves yield a closed subvariety of the Chow variety. Néron-Severi's theorem is presented as a result of the investigation of the cone of curves. The chapter culminates in Mori's "bend-and-break" technique which is applied to show that for a projective variety X containing a curve having negative intersection with the canonical class, there exists such rational curve as well. Other applications are vanishing results in positive characteristic obtained using a construction of Ekedahl. The final section of this chapter is devoted to smoothing of morphisms of curves and contains results for reduced curves which are "combs" with rational "teeth".

The next chapter III of the book aims to give a short illustration of the ideas of Mori's minimal model program in some special cases. It contains a proof of the cone theorem following Mori's original arguments. There is a section devoted to the case of surfaces which presents the classical theory from the viewpoint of the minimal model theory, accompanied by an introduction to del Pezzo surfaces X with $(K_X^2) \leq 4$ over an arbitrary field. The following chapters constitute the main part of the book: Chapter IV on "Rationally connected varieties" studies birational properties of varieties covered by rational curves. There are important parallels between the class of uniruled and rationally connected varieties, some of their numerical properties being stable under smooth deformations (at least if we exclude the case of characteristic > 0 ; this is necessary for some parts of this chapter). The following topics are covered in different sections:

Ruled and uniruled varieties;

Minimizing families of rational curves;

Rationally connected varieties;

Growing chains of rational curves;

Maximal rationally connected fibrations;

Rationally connected varieties over nonclosed fields.

Chapter V on Fano varieties considers the higher-dimensional generalizations of del Pezzo surfaces which are of increasing interest for Mori's program. Though the singular case is not considered, this should be important for a theory of Fano varieties with terminal singularities as well. There are various assertions to support the following principle: "The geometry of a Fano variety is governed by rational curves of low degree". It is explained how (in principle) a list of all Fano varieties in any given dimension could be obtained. Further, there is a proof of Mori's theorem characterizing the projective space as the only algebraic variety with ample tangent bundle. The final sections study concrete cases; especially section 5 is devoted to nonrational Fano varieties.

The appendix (chapter VI) gives a proof of Abhyankar's theorem which says that any birational morphism $X \rightarrow Y$ with Y smooth has only ruled exceptional divisors (without using resolution of indeterminacies). Further, there is an introduction to the intersection theory of divisors (following an unpublished paper of *S. Kleiman*) which is applied to obtain the asymptotic Riemann-Roch theorem for nef line bundles.

This book will be of continuing interest as well as a source of reference as for its presentation of the

material which is encouraging the reader to get through even with the technical demanding parts. It is (in a reasonable sense) self-contained and accompanied by many exercises.

Reviewer: [M.Roczen \(Berlin\)](#)

MSC:

- [14E30](#) Minimal model program (Mori theory, extremal rays)
- [14J26](#) Rational and ruled surfaces
- [14-02](#) Research exposition (monographs, survey articles) pertaining to algebraic geometry
- [14C05](#) Parametrization (Chow and Hilbert schemes)
- [14J45](#) Fano varieties
- [14M20](#) Rational and unirational varieties
- [14C40](#) Riemann-Roch theorems
- [14H10](#) Families, moduli of curves (algebraic)
- [14J10](#) Families, moduli, classification: algebraic theory

Cited in 21 Reviews Cited in 289 Documents

Keywords:

[Mori theory](#); [minimal models](#); [Fano variety](#); [Hilbert scheme](#); [rational curves](#); [Chow schemes](#); [vanishing](#); [positive characteristic](#); [Mori's minimal model program](#); [cone theorem](#); [del Pezzo surfaces](#); [Fano varieties](#)