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Geometric invariant theory and flips. (English) Zbl 0874.14042

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Author's introduction: Ever since the invention of geometric invariant theory, it has been understood that the quotient it constructs is not entirely canonical, but depends on a choice: the choice of a linearization of the group action. However, the founders of the subject never made a systematic study of this dependence. In light of its fundamental and elementary nature, this is a rather surprising gap, and this paper will attempt to fill it.

In one sense, the question can be answered almost completely. Roughly, the space of all possible linearizations is divided into finitely many polyhedral chambers within which the quotient is constant and when a wall between two chambers is crossed, the quotient undergoes a birational transformation which, under mild conditions, is a flip in the sense of Mori. Moreover, there are sheaves of ideals on the two quotients whose blow-ups are both isomorphic to a component of the fibred product of the two quotients over the quotient on the wall. Thus the two quotients are related by a blow-up followed by a blow-down.

The ideal sheaves cannot always be described very explicitly, but there is not much more to say in complete generality. To obtain more concrete results, we require smoothness, and certain conditions on the stabilizers which, though fairly strong, still include many interesting examples. The heart of the paper is devoted to describing the birational transformations between quotients as explicitly as possible under these hypotheses. In the best case the blow-ups turn out to be just the ordinary blow-ups of certain explicit smooth subvarieties, which themselves have the structure of projective bundles.

The last three sections of the paper put this theory into practice, using it to study moduli spaces of points on the line, parabolic bundles on curves, and Bradlow pairs. An important theme is that the structure of each individual quotient is illuminated by understanding the structure of the whole family. So even if there is one especially natural linearization, the problem is still interesting. Indeed, even if the linearization is unique, useful results can be produced by enlarging the variety on which the group acts, so as to create more linearizations. I believe that this problem is essentially elementary in nature, and I have striven to solve it using a minimum of technical machinery. For example, stability and semistability are distinguished as little as possible. Moreover, transcendental methods, choosing a maximal torus, and invoking the numerical criterion are completely avoided. The only technical tool relied on heavily is the marvelous Luna slice theorem. This theorem is used, for example, to give a new, easy proof of the Bialynicki-Birula decomposition theorem.

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MSC:

14L30	Group actions on varieties or schemes (quotients)
14L24	Geometric invariant theory
14E99	Birational geometry

Keywords:

geometric invariant theory; linearization of the group action; flip; blow-up; blow-down; birational transformations between quotients; Luna slice theorem

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