**Edmunds, David E.**; **Triebel, Hans**

**Function spaces, entropy numbers, differential operators.** (English) Zbl 0865.46020

Cambridge Tracts in Mathematics. 120. Cambridge: Cambridge Univ. Press. xi, 252 p. (1996).

This book, based on the results of the authors and their co-workers, deals with the following topics:

1) entropy numbers in quasi-Banach spaces,

2) distribution of eigenvalues of elliptic differential and pseudodifferential operators and

3) function spaces on $R^n$ and in domains.

The relationship between them is the main scope of the book. Two scales of function spaces are involved, the Besov-type spaces $B^s_{pq}$ and the Triebel-Lizorkin type spaces $F^s_{pq}$. Chapter 2 of the book deals with these function spaces. The reader can find here a condensed self-contained exposition of the basics of the theory together with some recent developments, concerning, e.g. embeddings, limiting cases, spaces in domains and others. Logarithmic Sobolev spaces are to be specially mentioned. Chapter 3 and 4 deal with entropy and approximation numbers of compact embeddings. One of the main typical example is as follows. Let $B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)$ be a compact embedding which is guaranteed by the conditions: $s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2})_+ > 0$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$, and let $e_k$ be the entropy numbers of this embedding. Then

$$c_1 k^{-(s_1-s_2)\backslash n} \leq e_k \leq c_1 k^{-(s_1-s_2)\backslash n}.$$

This goes back to M. S. Birman and M. Z. Solomiak, who first developed results of such a type, e.g. for the embedding $W^s_p(\Omega) \to L_q(\Omega)$. In this connection the authors observe that they work purely within the frameworks of Fourier-analytic techniques. Chapter 5 contains applications of the results for the entropy and approximative numbers of compact embeddings to the problem of distribution of eigenvalues of degenerate elliptic differential and pseudodifferential operators. This is based on the results which go back to B. Carl's estimate $|\mu_k| \leq \sqrt{2}e_k$, $\mu_k$ and $e_k$ being respectively the eigenvalues and the entropy numbers of a compact operator (in a quasi-Banach space).

Presented in a clear, self contained and well organized manner, the book is easy to read, although it deals with rather complicated considerations. The contents are the following:

Ch. 1. The abstract background. 1.1. Introduction. 1.2. Spectral theory in quasi-Banach spaces. 1.3. Entropy numbers and approximation numbers.

Ch. 2. Function spaces. 2.1. Introduction. 2.2. The spaces $B^s_{pq}$ and $F^s_{pq}$ on $R^n$. 2.3. Special properties. 2.4. Hölder inequalities. 2.5. The spaces $B^s_{pq}$ and $F^s_{pq}$ on domains. 2.6. The spaces $L_p(\log L)_a$ and the logarithmic Sobolev spaces. 2.7. Limiting embeddings.

Ch. 3. Entropy and approximation numbers of embeddings. 3.1. Introduction. 3.2. The embedding of $\ell^m_p$ in $\ell^m_q$. 3.3. Embeddings between function spaces. 3.4. Limiting embeddings in spaces of Orlicz type. 3.5. Embeddings in non-smooth domains.

Ch. 4. Weighted function spaces and entropy numbers. 4.1. Introduction. 4.2. Weighted spaces. 4.3. Entropy numbers.

Ch. 5. Elliptic operators. 5.1. Introduction. 5.2. Elliptic operators in domains: non-limiting cases. 5.3. Elliptic operators in domains: limiting cases. 5.4. Elliptic operators in $R^n$.

Reviewer: S.G.Samko (Faro)

**MSC:**


46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

46E39 Sobolev (and similar kinds of) spaces of functions of discrete variables

46-02 Research exposition (monographs, survey articles) pertaining to functional analysis

47B06 Riesz operators; eigenvalue distributions; approximation numbers, $s$-numbers, Kolmogorov numbers, entropy numbers, etc. of operators

35P20 Asymptotic distributions of eigenvalues in context of PDEs





**Keywords:**

logarithmic Sobolev spaces; weighted function spaces; entropy numbers; quasi-Banach spaces; distribution of eigenvalues; elliptic differential and pseudodifferential operators; scales of function spaces; Besov-type spaces; Triebel-Lizorkin type spaces; approximation numbers; compact embeddings; Fourier-analytic techniques; Orlicz type