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Geometry of rational curves on varieties. (English) Zbl 0872.14007

Miyaoka, Yoichi et al., Geometry of higher dimensional algebraic varieties. Basel: Birkhäuser. DMV Semin. 26, 1-127 (1997).

These lecture notes provide an introduction to some fundamental aspects of the geometry and classification theory of higher-dimensional complex algebraic varieties. They are based on a course that the author taught at a joint seminar with Th. Peternell on the topic “Mori theory” [DMV Seminar, Oberwolfach, Germany (April 1995)]. In fact, the present work and *Th. Peternell*'s article “An introduction to the classification of higherdimensional complex varieties” (in the same volume; see following review) form a unit, in that they are complementary to each other with respect to the various methodical components of *S. Mori*'s “minimal model program” for the birational classification theory of complex varieties. Roughly speaking, the notes under review focus on the algebraic methods, within the minimal model program, whereas Th. Peternell's related lectures cover the more complex-analytic and applicational aspects of the subject.

More concretely, a crucial observation in Mori's approach is that the existence (and configuration) of rational curves in a projective variety has a strong influence on the birational geometry of that variety. Actually, rational curves in projective varieties form the main obstruction to applying the general theory of deformations and moduli. This fact is thoroughly discussed, for example, in the survey article “The structure of algebraic threefolds: an introduction to Mori's program” by *J. Kollár* [Bull. Am. Math. Soc., New Ser. 17, 211-273 (1987; Zbl 0649.14022)], and in the seminar report “Higher dimensional complex geometry” by *H. Clemens*, *J. Kollár* and *S. Mori* [in: “Higher dimensional complex geometry”, Summer Seminar, Univ. Utah 1987, Astérisque 166 (1988; Zbl 0689.14016)]. According to this fundamental significance, the aim of the notes under review is to give a comprehensive, detailed overview of various aspects of the study of rational curves in algebraic varieties, including the following topics:

(i) Techniques to detect rational curves in special (classes of) varieties;

(ii) Characterization of uniruled varieties in terms of their canonical bundles;

(iii) Generic semi-positivity of the cotangent bundle of a non-uniruled variety and the “abundance conjecture” for threefolds;

(iv) Decomposition of a given variety into the non-uniruled part and the “rationally connected part”;

(v) Application of these techniques to the study of Fano varieties.

This covers most of the strategical steps of the “minimal model program”, while the complementary part (i.e., construction of minimal models by successively contracting rational curves and applying the “flip procedure”) is treated in Th. Peternell's complementary lectures.

The text is subdivided into five main lectures, each of which consists of several sections. The beautiful introduction to the article is entitled “Why rational curves?” and gives a lot of motivation, an explanation of Mori's strategy, and a list of the (really minimal) background material from basic algebraic geometry assumed throughout the treatise.

Lecture I discusses the procedure of producing rational curves in complete varieties via deformations of morphisms between varieties. The main result proved here is the existence theorem for rational curves in varieties, which is due to S. Mori and, in its more explicit form, to *Y. Miyaoka* and *S. Mori* (1986). In order to provide an almost self-contained exposition of this topic, the author gives a concise introduction to some fundamental concepts utilized in the course of the proof, such as Hilbert schemes, Chow schemes, deformations of morphisms, and the intersection pairing between curves and divisors on algebraic varieties.

Lecture II is devoted to the construction of non-trivial deformations of morphisms via reduction modulo primes. This technique is also due to S. Mori and is used to show the existence of rational curves on those smooth projective varieties whose canonical divisors are not numerically effective. This lecture also contains a proof of S. Mori's affirmative solution of the Hartshorne conjecture, stating that a smooth projective variety with ample tangent bundle must be a projective space, as well as the numerical characterization of uniruled varieties due to *Y. Miyaoka* and *S. Mori* [Ann. Math., II. Ser. 124, 65-69 (1986;

Zbl 0606.14030)].

Lecture III provides a refined characterization of uniruled varieties in terms of their tangent bundles. This is done by using quotient varieties by foliations in positive characteristic, which requires a detailed study of derivations and differential operators in this case. That as well as the transfer of the refined uniruledness criterion from positive characteristic to characteristic zero, which involves the theory of semi-stable sheaves and generically semi-positive sheaves, is explained in great detail. The chapter concludes with an equally detailed discussion of estimates for the second Chern class of a minimal variety, including a proof of the celebrated Bogomolov-Miyaoka inequality in the general case.

Lecture IV deals with the crucial "abundance conjecture" of *M. Reid* and *Y. Kawamata* and its affirmative answer in the case of minimal threefolds. After a quick introduction to the birational classification scheme in the sense of Enriques-Kodeira-Shafarevich-Iitaka, based on the concept of Kodaira dimension of algebraic varieties, the author presents the known facts concerning the general "abundance conjecture" and turns then to minimal complex threefolds. He gives full proofs of the non-negativity of the Kodeira dimension of those threefolds and of the correctness of the abundance conjecture for them. The results discussed here are essentially due to *Y. Kawamata* [see Invent. Math. 108, No. 2, 229-246 (1992; Zbl 0777.14011)] and the author himself [*Y. Miyaoka*, Compos. Math. 68, No. 2, 203-220 (1988; Zbl 0681.14019)].

The final lecture V is entitled "Rationally connected fibrations and applications". It is directed towards the study of the finer structure of uniruled varieties, beginning with a brief review of the classical birational theory of ruled surfaces for guiding motivation. A very thorough study of rationally connected varieties (after Kollár-Miyaoka-Mori) is followed by the proof of the existence theorem for maximal rationally connected fibrations (MRC-fibrations) of uniruled complex projective varieties, which is basically due to *Y. Miyaoka*, *S. Mori* and *F. Campana*. Furthermore, it is shown that rationally connected complex surfaces and threefolds are characterized by their global holomorphic differential forms, and that the existence of MRC-fibrations yields a classification of uniruled complex threefolds into three distinguished classes.

All this is done over the groundfield of complex numbers and makes use of complex-analytic methods. The concluding section of this lecture gives concrete applications of the preceding results to Fano manifolds. Using again reduction modulo prime numbers, the varieties considered here are defined over groundfields of characteristic p or zero. The main result of this section, which is again proved in full detail, states that the rational connectedness of a Fano manifold is equivalent to the triviality of its MRC-fibration. The proof is based upon the theory of relative deformations of morphisms, and the result by itself is lucidly put in the context of the already far-developed classification theory of smooth Fano threefolds.

Summing up, the lecture notes under review provide a brilliant, masterly written, very detailed and nearly self-contained introduction to the geometry of rational curves on algebraic varieties. Much more than that, the author gives a highly up-to-date report on the subject, which reflects the current state of the art in a very instructive and inspiring manner. This text is perhaps the best available introduction to the topics as a whole and, simultaneously, the perfect preparation for the study of the very recent, much more advanced and complete monograph "Rational curves on algebraic varieties" by *J. Kollár* (Berlin 1995).

For the entire collection see [Zbl 0865.14018].

Reviewer: W.Kleinert (Berlin)

MSC:

14E30 Minimal model program (Mori theory, extremal rays)
14H45 Special algebraic curves and curves of low genus
14D15 Formal methods and deformations in algebraic geometry
14M20 Rational and unirational varieties
14J45 Fano varieties
14J30 3-folds

Cited in **1** Review

Keywords:

abundance conjecture; minimal model program; Hartshorne conjecture; uniruled varieties; Bogomolov-Miyaoka inequality; minimal threefolds; fibrations; Fano manifolds; geometry of rational curves on algebraic varieties