Voevodsky, Vladimir
$\mathbb{A}^1$-homotopy theory. (English) Zbl 0907.19002
Doc. Math., Extra Vol. ICM Berlin 1998, vol. I, 579-604 (1998).

Whereas homological algebra has been a central ingredient in algebraic geometry for a very long time, the introduction of homotopical algebra, and in particular, stable homotopical algebra, is fairly recent. It was foreseen by various people that this might be a decisive step towards solving several open problems, and they have already been proven right. Most notably, it was a central ingredient in the author's proof of the Milnor conjecture [*V. Voevodsky*, "The Milnor conjecture", Max Planck Inst. preprint series or http://www.math.uiuc.edu/K-theory/0170/index.html (1996)]. In this paper the author attempts to show that the basic ideas are quite natural, and tries to make the reader feel at ease with the theory. In this he succeeds, perhaps at the risk of hiding the many difficult choices always being taken when constructing important theories.

The name "$\mathbb{A}^1$-homotopy theory" is meant to signify that the theory does for algebraic varieties and schemes what classical homotopy theory does for topological spaces; simply by using the affine line $\mathbb{A}^1$ in the rôle usually played by the unit interval.

The "spaces" in this theory are the smooth schemes over a Noetherian scheme S. However, in order to have arbitrary colimits, one considers sheaves in the Nisnevich topology instead, and the author manages to make this seem like a quite natural thing to do. From there the paper explains how one must proceed if one wants to study the unstable and stable homotopy categories in much the same spirit standard textbooks explain algebraic topology.

In the same way as the realization functor from simplicial sets to topological spaces may be constructed by modeling on the standard topological simplices, there is a realization functor $|-|_S$ from simplicial sets to "spaces". This allows one to "import" homotopy theory from simplicial sets, leading to a closed model category structure in the sense of *D. G. Quillen* ["Homotopical algebra", Lect. Notes Math. 43 (1967; Zbl 0168.20903)]. This relies on joint work with *F. Morel* and the author [$\mathbb{A}^1$-homotopy theory of schemes, Preprint 1998]. Hence one can localize at the weak equivalences and get an unstable homotopy category $H^{\mathbb{A}^1}$.

When one wants to stabilize, there is a twist. In addition to the simplicial circle $S^1_s = |\Delta^1/\partial\Delta^1|_S$ there is another circle $S^1_t = (\mathbb{A}^1 - \{0\}, 1)$, called the Tate circle, defined as the "space" $\mathbb{A}^1 - \{0\}$ pointed by 1. The surprising thing is that neither of these circles are enough to develop the theory. When we want to stabilize, we should do this by smashing with $T = (\mathbb{P}^1, \infty) \cong S^1_s \wedge S^1_t$. After discussing a Spanier-Whitehead approach to stabilization, the author considers the category of T-spectra. As always, there is an issue about the smash product, which the author bypasses by using the notion of symmetric spectra of *M. Hovey, B. Shipley* and *J. Smith* [http://hopf.math.purdue.edu/pub/Hovey-Shipley-Smith/ (1998)].

The paper then turns to applications. The first is motivic cohomology. If one wants to pursue the analogy with topology, one must construct spectra playing the rôle of the Eilenberg-Mac Lane spectra. The author explains why the most naïve approach does not work. The solution suggested by Suslin and developed jointly with the current author [*A. Suslin* and *V. Voevodsky*, "Singular homology of abstract algebraic varieties", Invent. Math. 123, No. 1, 61-94 (1996; Zbl 0896.55002)], is an analogy with the Dold-Thom theorem [*A. Dold* and *R. Thom*, "Quasifaserungen und unendliche symmetrische Produkte", Ann. Math., II. Ser. 67, 239-281 (1958; Zbl 0091.37102)] which constructs Eilenberg-Mac Lane spaces by means of the infinite symmetric product, giving for instance $K(\mathbb{Z}, n) \simeq \mathrm{Symm}^\infty(S^n)$. According to the author, showing that the resulting Eilenberg-Mac Lane spaces behave correctly (the analog of $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n+1)$ in ordinary homotopy theory), is the most difficult point of the theory, and has yet to find a smooth proof. At present a proof is available for S a smooth variety over a field of characteristic zero, and this relies on techniques developed in [*E. M. Friedlander* and *V. Voevodsky*, "Bivariant cycle cohomology", http://www.uiuc.edu/K-theory/075/5indexhtml(1995)] which in turn relies on Hironaka's resolution of singularities. The motivic cohomology is defined as the analog of singular cohomology. For smooth varieties over a field of characteristic zero, the motivic cohomology groups are shown to coincide with *S. Bloch*'s higher Chow groups ["Algebraic cycles and higher K-theory", Adv. Math. 61, 267-304

(1986; Zbl 0608.14004)].

The second application is to algebraic K-theory, and a construction is given which coincides with *Ch. Weibel*'s ["Homotopy algebraic K-theory", in: Algebraic K-theory and algebraic number theory, Contemp. Math. 83, 461-488 (1989; Zbl 0669.18007)]. This has close connections to the much older ideas present in *M. Karoubi* and *O. Villamayor*'s paper ["K-théorie algébrique et K-théorie topologique. I", Math. Scand. 28, 265-307 (1971; Zbl 0231.18018)] which contains elements with a strong resemblance to $\mathbb{A}^1$-homotopy theory.

The third application is to algebraic cobordism. This application is of special interest, due to its key rôle in the author's proof of the Milnor conjecture [loc. cit.]. The author ends the discussion by presenting a conjecture describing a part of algebraic cobordism in terms of the usual complex cobordism ring MU^*.

Reviewer: Bjørn Dundas (Trondheim)

MSC:

19E15	Algebraic cycles and motivic cohomology (K-theoretic aspects)
55P42	Stable homotopy theory, spectra
14F20	Étale and other Grothendieck topologies and (co)homologies
18G55	Nonabelian homotopical algebra (MSC2010)
14C35	Applications of methods of algebraic K-theory in algebraic geometry

Keywords:

homotopy theory for schemes; motivic cohomology; algebraic cobordism; algebraic K-theory; Milnor conjecture

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