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Hausdorff's theorem for posets that satisfy the finite antichain property. (English)

Zbl 0934.06005
Fundam. Math. 159, No. 1, 51-69 (1999).

A poset P is well-founded iff it has no infinite decreasing sequence; P satisfies the condition FAC iff every antichain of P is finite; P is scattered iff it has no subset which has the order-type of the set of rational numbers with their natural order. *F. Hausdorff* [Grundzüge der Megenlehre (Veit & Comp., Leipzig) (1914; JFM 45.0123.01), Kapitel IV, §6] had considered this concept only for linearly ordered sets. If P and Q are posets with the same carrier sets and orders $\leq_P$ resp. $\leq_Q$, then Q augments P, iff $\leq_Q$ contains $\leq_P$. The set of all possible augmentations of P is denoted by $\operatorname{aug}(P)$.

When P satisfies the FAC, $(\mathcal{A}(P), \supset)$ is the poset of all nonempty antichains of P under inverse inclusion. This set is well-founded and thus it has a rank function $\operatorname{rk}_{\mathcal{A}}$. The antichain rank of P is then its image $\operatorname{rk}_{\mathcal{A}}(P)$, which is an ordinal. The main theorem then states the following: Let ρ be an ordinal and let $\operatorname{aug}(\mathcal{H}^\rho)$ be the closure of the class of all well-founded posets with antichain rank $\leq \rho$ under inversion, lexicographic sums, and augmentation. Then it contains the class of all scattered FAC-posets with rank $\leq \rho$. So $\operatorname{aug}(\mathcal{H})$, which is the closure of the well-founded posets with FAC under inversion, lexicographic sums, and augmentation, is the class of all scattered FAC-sets. For $\rho = 1$ this implies Hausdorff's theorem [loc. cit.]. In the proof, the authors introduce a new product operation for ordinals, which they call the Hessenberg based product.

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MSC:

06A06 Partial orders, general
03E10 Ordinal and cardinal numbers

Keywords:

scattered sets; finite antichain property; antichain rank; well-founded posets; product operation for ordinals

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