Knutson, Allen; Tao, Terence

The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. I: Proof of the saturation conjecture.

(English) Zbl 0944.05097
J. Am. Math. Soc. 12, No. 4, 1055-1090 (1999).

This is a remarkable paper which is devoted to the final solution of the following old and fundamental combinatorial problem: For given dominant weights λ, μ, ν and the corresponding irreducible $\mathrm{GL}_n(\mathbb{C})$-modules V_λ, V_μ, V_ν, does the tensor product $V_\lambda \otimes V_\mu \otimes V_\nu$ contain a $\mathrm{GL}_n(\mathbb{C})$-invariant vector? Without essential lost of generality, replacing ν with its dual ν^*, another standard formulation of the problem is for which polynomial modules $V_\lambda, V_\mu, V_{\nu^*}$ is the coefficient $c^{\nu^*}_{\lambda,\mu}$ in the Littlewood-Richardson rule $V_\lambda \otimes V_\mu \cong \sum c^{\nu^*}_{\lambda,\mu} V_{\nu^*}$ is different from 0? The answer to the problem under consideration is known asymptotically. *A. A. Klyachko* [Sel. Math., New Ser. 4, No. 3, 419–445 (1998; Zbl 0915.14010)] gave a finite set of linear inequalities derived from the Schubert calculus and λ, μ, ν satisfying this system if and only if there exists a positive integer N such that $V_{N\lambda} \otimes V_{N\mu} \otimes V_{N\nu}$ contains a $\mathrm{GL}_n(\mathbb{C})$-invariant vector. Since the set of triples (λ, μ, ν) corresponding to a $\mathrm{GL}_n(\mathbb{C})$-invariant vector form an additive monoid, the complete solution of the problem for the description of such triples depends on the saturation of the monoid and this is the main result of the paper under review: If $(V_{N\lambda} \otimes V_{N\mu} \otimes V_{N\nu})^{\mathrm{GL}_n(\mathbb{C})} > 0$ for some $N > 0$, then $(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n(\mathbb{C})} > 0$. The main tool for the proof is the Berenstein-Zelevinsky polytope associated to the triple (λ, μ, ν) in which the number of lattice points is the corresponding Littlewood-Richardson coefficient. The authors give a new description using their honeycomb model.

As an immediate consequence the authors obtain an affirmative answer to the Horn conjecture from 1962 which gives a recursive system of inequalities reducing the problem to lower-dimensional Littlewood-Richardson questions. The considered problem is also related to many other classical problems. It turns out that they all have the same answer and are consequences of the result by Klyachko and the saturation conjecture. In particular, the authors give a new proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture for $\mathrm{GL}_n(\mathbb{C})$ which states that if $w\lambda + v\nu$ is in the positive Weyl chamber, then $V_{w\lambda+v\nu}$ is a constituent of $V_\lambda \otimes V_\nu$ (and is known to be true for all Lie groups). Another application is to Hermitian matrices solving a problem with more than 100 years of history: If λ, μ, ν are sequences of n descending real numbers, do there exist Hermitian matrices A and B with eigenvalues λ and μ, respectively, such that $A + B$ has eigenvalue ν?

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MSC:

05E05	Symmetric functions and generalizations
20G05	Representation theory for linear algebraic groups
15A42	Inequalities involving eigenvalues and eigenvectors
05E15	Combinatorial aspects of groups and algebras (MSC2010)
22E46	Semisimple Lie groups and their representations
05E10	Combinatorial aspects of representation theory
14M15	Grassmannians, Schubert varieties, flag manifolds
52B20	Lattice polytopes in convex geometry (including relations with commutative algebra and algebraic geometry)

Keywords:

honeycombs; symmetric functions; Littlewood-Richardson rule; Berenstein-Zelevinsky patterns; Horn's conjecture; saturation; Klyachko inequalities; Weyl chamber; Hermitian matrices

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