Le Gall, Jean-François

Spatial branching processes, random snakes and partial differential equations. (English)

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Lectures in Mathematics, ETH Zürich. Basel: Birkhäuser. viii, 162 p. (1999).

These lecture notes provide a concise and essentially self-contained monograph on the theory of superprocesses (i.e. measure-valued branching processes), a topic that has been intensively studied during the last decade. The presentation focuses on the approach based on the so-called Brownian snake, a path-valued process which has been introduced by the author and then further studied by himself and his collaborators. The Brownian snake both yields a simple construction of superprocesses with a quadratic branching mechanism, and gives new insights on their connections with certain partial differential equations.

The first chapter presents an appealing non-technical overview of the text, which is very useful to grasp right from the beginning the main trends of the monograph. The second chapter introduces basic features on continuous-state branching processes (including a construction as the limit of continuous-time Galton-Watson processes), and on superprocesses (which combine the branching mechanism with a spatial motion). The key fact that Brownian excursions can be used to code the genealogy of continuous-state branching processes with quadratic branching mechanism is developed in the third chapter. More precisely, there is in the discrete setting a well-known observation due to Harris according to which the excursion away from 0 of a simple random walk can be viewed as the contour process of a Galton-Watson tree with geometric offspring distribution. In the continuous setting, this underlies a representation of certain random trees (including Aldous' continuum random tree) in terms of the Brownian excursion.

The Brownian snake W is constructed in the fourth chapter. Roughly, one first introduces a reflecting Brownian motion $(\zeta_s, s \geq 0)$ which serves as the lifetime process, in the sense that for each $s \geq 0$, W_s is path-valued in some Polish space E and ζ_s represents its lifetime. Conditionally on the lifetime process ζ, W is a time-inhomogeneous Markov process, whose evolution can be described informally as follows: when the lifetime ζ_s decreases, the path W_s is erased from its tip, whereas when ζ_s increases, W_s is extended from its tip according to some spatial motion on E. The tree structure embedded in the Brownian excursion is then the key to the remarkable construction of superprocesses with quadratic branching mechanism as the occupation measure of the tip of the snake, $\hat{W}_s = W(\zeta_s)$. This construction is then used for instance to investigate the dimension of the support of certain superprocesses.

The next three chapters (V, VI and VII) are devoted to connections with the nonlinear PDE $\Delta u = 4u^2$. This corresponds to the special case when $E = \mathbb{R}^d$ and the spatial motion is given by a Brownian motion. First, the so-called exit measure is constructed by stopping each path W_s at its exit time from some given domain D. The Laplace transform of the exit measure provides a solution to the analogue of the Dirichlet problem in D, which consists in finding a function u on D that solves $\Delta u = 4u^2$ and has given smooth limit values at the boundary ∂D. This is used to investigate extremal solutions that blow up at the boundary, to characterize polar sets for superprocesses and to provide a probabilistic representation of all the positive solutions in dimension $d = 2$ when D is the unit disk. Finally, the last chapter is based on recent joint works with Y. Le Jan. The main purpose is to propose a construction of superprocesses with a general branching mechanism ψ, using an analogue of the Brownian snake. The key here is given by the Lévy process Y with only positive jumps that has Laplace exponent ψ (a first connection between Y and the continuous state branching process with mechanism ψ has been pointed out by Lamperti in the mid-60's). The so-called height process H associated with Y provides a natural coding for the branching mechanism ψ, and the generalized snake obtained by replacing the reflecting Brownian motion by H in the original construction can be used to represent superprocesses with branching mechanism ψ.

In conclusion, this is a very accessible text on superprocesses, written by a leading expert of the field. In slightly less than 150 pages, it provides a clear and precise presentation of several important aspects of the theory of superprocesses developed over the recent years. There is no doubt that such a monograph will be used both by beginners to learn the theory and by experts as a reference text.

Reviewer: J.Bertoin (Paris)

MSC:

60-02 Research exposition (monographs, survey articles) pertaining to probability theory
60J85 Applications of branching processes
60K35 Interacting random processes; statistical mechanics type models; percolation theory
60H15 Stochastic partial differential equations (aspects of stochastic analysis)
60J80 Branching processes (Galton-Watson, birth-and-death, etc.)

Keywords:

superprocesses; Brownian snake; branching processes; nonlinear partial differential equations