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Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. (English) Zbl 0956.42014

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There is a revolution sweeping through asymptotics of orthogonal polynomials called the Riemann-Hilbert method. It has enabled researchers such as the present authors and Deift, Kriecherbauer, MacLaughlin and others to obtain very precise (and uniform) asymptotics for orthogonal polynomials for exponential weights, in situations where the classical Bernstein-Szegö methods give limited precision. And this paper is the record of one of the first breakthroughs in this exciting development.

Let

$$V(z) := gz^4/4 + tz^2/2,$$

where $g > 0 > t$, so that V is a double-well potential. Let $0 < \varepsilon < 1$, and for $n \geq 1$ consider a parameter N satisfying

$$\varepsilon > \frac{n}{N} < \frac{t^2}{4g} - \varepsilon.$$

Let us consider the monic orthogonal polynomials P_n with respect to the varying weight $w := \exp(-NV)$, so that

$$\int_{-\infty}^{\infty} P_n P_m \exp(-NV) = h_n \delta_{mn},$$

where $h_n > 0$. The authors establish very precise asymptotics for P_n and the associated recurrence coefficients as $n \to \infty$. Then they apply these to establish universality of the local distribution of eigenvalues in the matrix model with quartic potential.

A key point in the analysis is the Fokas-Its-Kitaev Riemann-Hilbert problem, in which the orthogonal polynomials appear explicitly. This is followed by use of an approximate solution to the Riemann-Hilbert problem, and a proof that the approximate solution gives the asymptotic formula. The paper contains an extensive review of related literature; in particular, the context of the results and their motivation is very clearly presented. This paper will be of great use to anyone interested in orthogonal polynomials and their applications.

Reviewer: D.S.Lubinsky (Wits)

MSC:

42C05	Orthogonal functions and polynomials, general theory of nontrigonometric harmonic analysis
33C05	Classical hypergeometric functions, $_2F_1$
15B52	Random matrices (algebraic aspects)
33C45	Orthogonal polynomials and functions of hypergeometric type (Jacobi, Laguerre, Hermite, Askey scheme, etc.)
41A60	Asymptotic approximations, asymptotic expansions (steepest descent, etc.)

Keywords:
random matrices; asymptotics; orthogonal polynomials; exponential weights; Riemann-Hilbert problem

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