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Notes on Seiberg-Witten theory. (English) Zbl 0978.57027

Graduate Studies in Mathematics. 28. Providence, RI: American Mathematical Society (AMS). xviii, 484 p. (2000).

This volume provides a general introduction to Seiberg-Witten theory, which covers many technical issues involved in the definition and main properties of Seiberg-Witten invariants on both 4-dimensional and 3-dimensional manifolds.

The book is organized in the following way. The first chapter contains general background material on bundles and characteristic classes, elliptic operators, Clifford algebras, spinors and Dirac operators, complex geometry, and Fredholm theory. In the second chapter the author introduces Seiberg-Witten invariants of four-manifolds and describes the main properties of the moduli spaces: compactness, local structure, smoothness, orientability. The third chapter contains an overview of some of the early successes of Seiberg-Witten theory: Witten's description of Seiberg-Witten monopoles on algebraic surfaces in terms of vortices, which in turn admit an interpretation in terms of divisors, with subsequent applications, such as non-triviality of the invariants for Kähler surfaces, the invariance under diffeomorphisms of the canonical class, and the smooth classification of elliptic surfaces. The same chapter contains an application to the failure of the h-cobordism theorem in 4-dimensions, and a discussion of the Taubes result on the nontriviality of Seiberg-Witten invariants for symplectic 4-manifolds. The remaining part of the book (chapter 4) is dedicated to the so-called cut-and-paste problem. This occupies a substantial part of the book, as the technical issues involved are very demanding. The idea underlying the cut-and-paste problem arises in Atiyah's axiomatization of topological quantum field theory [*M. Atiyah*, Proc. Symp. Pure Math. 48, 285-299 (1988; Zbl 0667.57018)]. Given a decomposition of a compact 4-manifold given by a separating 3-manifold, the cut-and-paste technique consists of the study of moduli spaces on the resulting 4-manifolds with boundary completed with infinite cylindrical ends, the moduli space of monopoles on the 3-manifold, and their relation to the moduli space on the compact 4-manifold. The technical difficulties in this kind of problem revolve around a few main issues: the correct setup for treating moduli spaces on noncompact manifolds, the identification of limits of monopoles in the process of stretching a long cylindrical region around the separating 3-manifold in the compact 4-manifold, the issue of which pairs of solutions on the two 4-manifolds with infinite cylindrical ends can be glued together to form a monopole on the compact 4-manifold with a long cylinder. These issues are treated in the text by first considering the local and global properties of moduli spaces of finite energy monopoles on 4-manifolds with infinite cylindrical ends. The main issues discussed are the Cappell-Lee-Miller gluing theorem, the computation of the virtual dimension, and then smoothness, compactness, and orientation, which involve more delicate arguments than the corresponding results on compact manifolds. Then the convergence and gluing of monopoles is discussed, following the lines of the very general gluing theorem of Mrowka's thesis, by first proving a local gluing theorem, then dealing with the surjectivity issue using Lojasewicz's inequality, and then proving global gluing theorems (possibly in the presence of obstructions). The author's treatment however differs from other treatments of gluing theory in the fact that it derives spectral estimates and Mayer-Vietoris type local model directly from the Cappell-Lee-Miller theorem.

There are other currently available accounts of Seiberg-Witten theory [*J. W.Morgan*, The Seiberg-Witten equations and applications to the topology of smooth four-manifolds, Math. Notes Princeton 44 (1996; Zbl 0846.57001); *J. D. Moore*, Lectures on Seiberg-Witten invariants, Lect. Notes Math. 1629 (1996; Zbl 0857.58001); *M. Marcolli*, Seiberg-Witten gauge theory, Texts Read. Math. 17 (1999; Zbl 0954.53003)]. Though some overlap with other references in this list is inevitable, the volume under review presents a different perspective, especially in the part on gluing theory, and the detailed account makes it suitable for a graduate textbook on the subject.

Reviewer: Matilde Marcolli (Cambridge /MA)

MSC:

57R57	Applications of global analysis to structures on manifolds
57-02	Research exposition (monographs, survey articles) pertaining to manifolds and cell complexes
57R19	Algebraic topology on manifolds and differential topology
57R15	Specialized structures on manifolds (spin manifolds, framed manifolds, etc.)
58D27	Moduli problems for differential geometric structures
14J80	Topology of surfaces (Donaldson polynomials, Seiberg-Witten invariants)
53C55	Global differential geometry of Hermitian and Kählerian manifolds
58J05	Elliptic equations on manifolds, general theory
58J52	Determinants and determinant bundles, analytic torsion

Keywords:

Seiberg-Witten gauge theory; moduli spaces; gluing techniques; glueing; Seiberg-Witten invariants; cut-and-paste problem