Conrad, Brian
Grothendieck duality and base change. (English) Zbl 0992.14001
Lecture Notes in Mathematics. 1750. Berlin: Springer. x, 296 p. (2000).

Grothendieck duality theory on noetherian schemes plays a crucial role in various branches of algebraic and arithmetic geometry, ranging from the study of moduli spaces of algebraic curves up to the arithmetic theory of modular forms. About fourty years ago, *A. Grothendieck* initiated this theory, the main goal of which was to produce a certain "trace map" in the cohomology theory of coherent sheaves generalizing the classical Serre duality for smooth schemes over a field. The foundations of what is now called Grothendieck duality theory are worked out in *R. Hartshorne*'s celebrated monograph "Residues and duality", Lect. Notes Math. 20 (1966; Zbl 0212.26101) published more than thirty-five years ago.

The foundational framework developed in Hartshorne's book, based on residual complexes and the notion of a dualizing sheaf, makes this duality theory quite computable in terms of differential forms and residues, and this efficient computability has turned out to be extremely useful in many concrete situations in both algebraic and arithmetic geometry.

However, in Hartshorne's construction of Grothendieck duality theory there are some assumptions on compatibility conditions, together with some explications of abstract results, which are not rigorously proven and are, in fact, quite difficult to verify. The hardest compatibiliy condition in the theory, and also one of the most important, is the base change compatibility of the trace map in the case of proper Cohen-Macaulay morphisms with pure relative dimension. For example, this important special case occurs in the study of flat families of semi-stable curves and their moduli. Although there are simpler methods for obtaining duality theorems in the projective Cohen-Macaulay case, which allow to ignore the base change compatibility problem [*A. Altman* and *S. Kleiman*, "Introduction to Grothendieck duality theory", Lect. Notes Math. 146 (1970; Zbl 0215.37201)], there remains the fundamental question of whether the hard unproven compatibilities in the foundations elaborated by R. Hartshorne can really be verified.

The aim of the book under review is to give an affirmative answer to this long-standing problem, i.e., to provide rigorous proofs of those compatibility theorems, and to derive some important consequences and examples of this (finally established) abstract theory. In this vein, and as the author himself points out, the present book should therefore be viewed as a companion (and complement) to R. Hartshorne's classical monograph from 1966. Also, it is by no means a logically independent treatment of Grothendieck duality theory from the very beginning, as it actually (and often) appeals to results proven in Hartshorne's standard book.

As to the contents, the book under review consists of five chapters and two appendices.

Chapter 1, the introduction, provides a first overview, some motivation, and the definitions of most of the basic constructions in Hartshorne's approach to Grothendieck duality theory.

Chapter 2, entitled "Basic compatibilities", is concerned with verifying several important compatibility conditions underlying Hartshorne's approach. The basic functorial formalism needed to this end is developed and discussed in full detail.

Chapter 3 comes with the title "Duality foundations" and is devoted to a thorough discussion of Grothendieck's notion of a residual complex. The material covered here includes, among other topics, dualizing complexes, residual complexes, the general trace map, Grothendieck-Serre duality, dualizing sheaves, Cohen-Macaulay maps, and the general base change theory for dualizing sheaves.

Chapter 4 culminates in the proof of the general duality theorem for proper Cohen-Macaulay maps with pure relative dimension between noetherian schemes admitting a dualizing complex. This result, the proof of which is indeed rather involved and profound, completes Hartshorne's approach to Grothendieck duality theory and, relievingly, justifies it in a concluding manner.

Also, the author compares his result to the (classical) duality theorem of Verdier [in: *J.-L. Verdier*, Algebr. Geom., Bombay Colloq. 1968, 393-408 (1969; Zbl 0202.19902)] in a very enlightening manner.

Chapter 5, simply entitled "Examples" makes the abstract derived category duality theorem (theorem 4.3.1. in the present book) somewhat concrete. This is done by recovering from the general theory (developed here) some of the most widely used consequences for duality of higher direct image sheaves and, in the second part, by deducing the classical results of M. Rosenlicht that describe the dualizing sheaf and the trace map on a reduced proper curve over an algebraically closed field in terms of the so-called regular differentials and residues.

Appendix A addresses the topic of the residue symbol utilized in R. Hartshorn's standard text on Grothendieck duality theory. After stating the main results on residues and cohomology with supports, the author provides full and detailed proofs for them.

Appendix B deals particularly with the theory of residues for, and the trace map on smooth curves. The discussion given here makes some of the corresponding results contained in Hartshorne's book more explicit and digestible, on the one hand, and throws the bridge to the related duality theory for Jacobian varieties, on the other hand.

Whenever appropriate in the course of the text, the author compares his approach to the different approach to duality by J. Lipman, which seems to be even more general and far-reaching, on the one hand, but which is even more abstract and "categorically" involved than the original approach by Grothendieck and Hartshorne [cf. *L. Alonso*, *A. Jeremias* and *J. Lipman*, "Studies in duality on noetherian formal schemes and non-noetherian ordinary schemes", Contemp. Math. 244 (1999; Zbl 0927.00024)], on the other hand.

The book under review provides, altogether, an important and major contribution towards a better understanding of Grothendieck duality theory in its full generality.

Reviewer: Werner Kleinert (Berlin)

MSC:

14A15	Schemes and morphisms	Cited in **2** Reviews
14F10	Differentials and other special sheaves; D-modules; Bernstein-Sato ideals and polynomials	Cited in **82** Documents

Keywords:

schemes; sheaves of differentials; dualizing sheaves; residues; duality theorems; base change theorems; Cohen-Macaulay maps; trace maps; algebraic curves; morphisms

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