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Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. (English) Zbl 0967.55010

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Cohomology theories defined on real oriented spaces and spectra were originally introduced by *M. F. Atiyah* [Q. J. Math., Oxford II. Ser. 17, 367-386 (1966; Zbl 0146.19101)] who defined a version of K-theory, $KR^*(\ )$, for spaces with the extra structure (a real orientation) of an action by $\mathbb{Z}/2$ generalizing that of complex conjugation on the complex points of a real algebraic variety. *P. S. Landweber* [Bull. Am. Math. Soc. 74, 271-274 (1968; Zbl 0181.26801)] and *S. Araki* [Jap. J. Math., New Ser. 5, 403-430 (1979; Zbl 0443.55003)] extend this, introducing real oriented cobordism, $M\mathbb{R}^*(\cdot)$, for such spaces and an analog, $BP\mathbb{R}^*(\cdot)$, of Brown-Peterson theory. The paper under review is devoted to setting up Adams-Novikov type spectral sequences for these theories.

The paper begins with a summary of the Landweber-Araki results, expressed in the language of modern equivariant stable homotopy theory. (Thus, coefficient rings of spectra such as $M\mathbb{R}$ are graded over $RO\mathbb{Z}/2$ rather than bigraded as in the original papers.) This is extended to give a construction of real oriented analogs (such as $BP\mathbb{R}(n)$, $E\mathbb{R}(n)$, $K\mathbb{R}$, and $K\mathbb{R}(n)$) of the standard complex oriented spectra derived from MU. It is shown that $K\mathbb{R}$ coincides with Atiyah's KR, and $K\mathbb{R}(n)_*$ is computed. The authors establish strong completion theorems for $BP\mathbb{R}$ and $M\mathbb{R}$, calculate the Hopf algebroid $(BP\mathbb{R}_*, BP\mathbb{R}_*BP\mathbb{R})$ and show that it is flat. The spectral sequence of the title is then defined as the Borel cohomology spectral sequence of a certain cosimplicial spectrum, $C(U, FU, FS^0)$, and the E_2 term is identified with $\mathrm{Ext}_{BP\mathbb{R}_*BP\mathbb{R}}(BP\mathbb{R}_*, BP\mathbb{R}_*)$ using flatness. The convergence to $\pi_*(S^0_{\mathbb{Z}/2})$ is a consequence of the completion theorem. Three further topics are developed, based on this construction. First a modification of this spectral sequence, using the idea of a graded spectrum, is set up which converges to the nonequivariant stable 2-stem. This is more subtle than simply the spectral sequence based on $BPO = BP\mathbb{R}^{\mathbb{Z}/2}$ because (BPO_*, BPO_*BPO) is not flat. There is some overlap in this section with recent results of M. Hopkins and H. Miller (to appear), and the authors describe these results from their point of view. Next the relation between the real oriented Adams-Novikov spectral sequence and the $\mathbb{Z}/2$ equivariant Adams spectral sequence for ordinary mod 2 cohomology is explored. (This involves a study of the $\mathbb{Z}/2$ equivariant Steenrod algebra, an area previously developed by J. Greenlees (PhD. thesis, Cambridge, 1985). Finally, a real oriented version of the Miller-Novikov spectral sequence is developed, i.e. a spectral sequence whose E_1 term is the E_2 term of a Cartan-Eilenberg spectral sequence associated to the $\mathbb{Z}/2$ equivariant Steenrod algebra and which converges to the E_2 term of the real oriented Adams-Novikov spectral sequence.

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MSC:

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55T99 Spectral sequences in algebraic topology

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