Mukai, Shigeru

An introduction to invariants and moduli. Transl. by W. M. Oxbury. (English) Zbl 1033.14008

Cambridge Tracts in Mathematics 81. Cambridge: Cambridge University Press (ISBN 0-521-80906-1/hbk). xx, 503 p. (2003).

This book is the English translation of *S. Mukai*'s two-volume text "Theory of moduli. I. II" published in 1998 and 2000 in Japanese. As the author points out in the preface, the aim of this text is to provide a concise introduction to both the geometry of algebraic varieties and to algebraic moduli theory. However, the main emphasis is put on algebraic moduli spaces, together with their allied aspects of classical and geometric invariant theory, while the necessary basic concepts of (elementary) algebraic geometry are developed along this guiding principle. Moreover, the text is intended to be of introductory character, and perhaps also of somewhat panoramic kind, which means it does not claim to be ultimately comprehensive, or even encyclopedic, with respect to the subject. For instance, as for the discussion of concrete moduli spaces in algebraic geometry, the author focuses on Jacobians of curves (as moduli spaces) and on moduli spaces of vector bundles over curves, together with some illustrating recent applications in mathematical physics (Verlinde formula), yet leaving aside the construction of moduli spaces for algebraie curves, algebraic surfaces, and principally polarized Abelian varieties. Instead, much effort is spent to explain the methodical framework of classical and modern invariant theory in great detail, and that with an admirable sense for the mathematical-cultural significance of this venerable discipline.

As to the contents of the book, the text comes with twelve chapters, each of which is subdivided into several sections. After a very motivating, appetizing and well-considered introduction to the concept of moduli, which also refers to the following single chapters of the book, the undermentioned materials are covered:

Chapter 1. Invariants and moduli: This chapter discusses an elementary parameter space for plane conics, invariants of groups and Hilbert series, classical binary invariants, the basics of affine and projective plane curves, and the classical facts on elliptic curves.

Chapter 2. Rings and polynomials: This chapter gives a brief account of the basic algebraic techniques which form the foundation of both invariant theory and algebraic geometry in general. Apart from these topics in commutative algebra, the author begins the treatment of rings of invariants, culminating in the discussion of Nagata's counter-example to the fourteenth Hilbert problem.

Chapter 3. Algebraic varieties: This is a very concise introduction to basic abstract algebraic geometry (varieties, no schemes!), incorporating also a little bit on algebraic groups and toric varieties.

Chapter 4. Algebraic groups and rings of invariants: This chapter is on classical invariant theory, including representations of algebraic groups, Lie spaces of algebraic groups and the Casimir operator, Hilbert's finiteness theorem for rings of invariants with respect to linearly reductive groups, the Cayley-Sylvester counting theorem, and the proof of the geometric reductivity of the group $\mathrm{SL}(2,k)$.

Chapter 5. The construction of quotient varieties: This chapter deals with affine quotients modulo reductive algebraic group actions and with the concept of stability in this context. This general theory is then applied to the classical case of hypersurfaces of degree d in projective spaces. The construction of their moduli spaces, nullforms, and the projective quotient map concludes this chapter.

Chapter 6. The projective quotient: Quotients of projective varieties by PGL(N) are the subject of study in this chapter. Instead of working in the general framework of Mumford's geometric invariant theory, the author takes a pleasant and very efficient shortcut by constructing the so-called "Proj quotients" of affine varieties modulo GL(N).

Chapter 7. The numerical criterion and some applications: This chapter discusses the important concept of (semi-)stability for points in an affine variety acted on by a linearly reductive group. The main topic here is the Hilbert-Mumford numerical criterion of stability, together with concrete applications to the cases of projective hypersurfaces, cubic surfaces, and finite point sets in projective space.

Chapter 8. Grassmannians and vector bundles: The author introduces here Grassmannian varieties and such related constructions as Hilbert series, standard monomials, Young tableaux, the Grassmann functor, and the representation of Grassmannians as quotient varieties. Along this path, further fundamental concepts of algebraic geometry are touched upon, including locally free modules over a ring, the Picard group of a ring, and vector bundles over algebraic varieties. This will be needed in the sequel, when moduli spaces of vector bundles over curves are described.

Chapter 9. Curves and their Jacobians: Here algebraic curves make their entry, and the author develops their classical theory in a very efficient, elegant and succinct way, including their Jacobians and Picard varieties from the viewpoint of moduli of line bundles on them.

Chapter 10. Stable vector bundles on curves: In this chapter, the author extends some essential parts of the line bundle theory of the preceding chapter to higher-rank vector bundles over curves, and then he constructs the moduli space of rank-2 vector bundles. The notion of stability appears here in a very natural, crucial and instructive way.

Chapter 11. Moduli functors: The results of chapters 9 and 10 are here reconsidered and conceptually generalized. The author introduces the categorical framework of moduli theory, in general, discusses fine and coarse moduli spaces, the Picard functor, Poincaré bundles, and reviews the already constructed moduli spaces in this general context. In the course of this chapter, many other concrete examples of moduli spaces are also sketched.

Chapter 12. Intersection numbers and the Verlinde formula: In this final chapter, the author gives, as an important contemporary application of moduli theory, a beautiful treatment of the celebrated Verlinde formula for rank-2 vector bundles on curves. The proof of the Verlinde formula presented here is that of D. Zagier (1995), making use of the enumerative formulae for moduli spaces of vector bundles due to M. Thaddeus (1992). Besides, this chapter touches upon various related algebro-geometric notions and methods such as the diverse Riemann-Roch theorems, the Mumford relations in the enumerative geometry of moduli spaces, and the theory of quasi-parabolic vector bundles (à la Mehta-Seshadri).

Each chapter comes with its own bibliography, but the book does not contain any exercises for the reader.

All together, this is a marvellous and masterly introduction to moduli theory and its allied invariant theory. The exposition fascinates by great originality, glaring expertise, art of easiness and lucidity, up-to-dateness, reader-friendliness, and power of inspiration. As for further reading, the reader is recommended to consult the book "Advances in moduli theory" by *Y. Shimizu* and *K. Ueno* [Transl. Math. Monographs. 206 (2002; Zbl 0987.14001), which was in fact a companion volume to the present book when published in Japanese. Also, the recently published text "Lectures on invariant theory" by *I. V. Dolgachev* [Lond. Math. Soc. Lect. Note Ser. 296 (2003; Zbl 1023.13006)] should be regarded as a very useful companion book for parallel reading. Actually, it is very gratifying to see the textbook literature on invariant theory and moduli theory finally increasing, and S. Mukai's book under review is certainly an outstanding contribution to this development.

Reviewer: Werner Kleinert (Berlin)

MSC:

14D20	Algebraic moduli problems, moduli of vector bundles
14L24	Geometric invariant theory
14H60	Vector bundles on curves and their moduli
14H40	Jacobians, Prym varieties
14M17	Homogeneous spaces and generalizations
13A50	Actions of groups on commutative rings; invariant theory
14-01	Introductory exposition (textbooks, tutorial papers, etc.) pertaining to algebraic geometry
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry

Keywords:

geometric invariant theory; quotient varieties; Picard varieties; stability of vector bundles; Verlinde formula; moduli spaces; Jacobians of curves; fourteenth Hilbert problem; line bundle