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Let \( A \) denote the set of all real-valued arithmetic functions \( f \), \( P \) the set of all \( f \in A \) such that \( f(1) \) is positive, and \( M \) the set of all real multiplicative \( f \). Let \( \ast \) denote the Dirichlet product and \( \times \) the unitary product. It is shown that the groups \( \{ P, \ast \} \), \( \{ M, \ast \} \), \( \{ P, X \} \), \( \{ M, X \} \) and \( \{ A, + \} \) are all isomorphic. The proofs are based on properties of a logarithm operator \( L : P \to A \) defined (in the Dirichlet case) by

\[
Lf(n) = \sum_{d \mid n} f(d) f^{-1}(n/d) \log d \quad \text{if } n > 1, \quad Lf(1) = \log f(1).
\]

It is shown that \( L \) possesses the logarithmic property \( L(f \ast g) = L(f) + L(g) \) and is a bijection of \( P \) onto \( A \). An analogous result holds in the unitary case. If \( f \in P \) and \( r \) is a real number, we define the \( r \)th power of \( f \) by \( f^r = E(rL_f) \), where \( E = L^{-1} \). It is shown that \( f \) is multiplicative if and only if \( LF(n) = 0 \) whenever \( n \) is not a prime power. It follows that \( f^r \) is multiplicative whenever \( f \) is. Trigonometric operators are constructed from \( E \) as in ordinary analysis, and their properties lead to further isomorphism theorems. Finally, an extension of these results to complex-valued arithmetic functions is indicated.

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For a scan of this review see the web version.

MSC:

11A25  Arithmetic functions; related numbers; inversion formulas

Keywords:

real-valued arithmetic functions; Dirichlet product; unitary product; isomorphism theorems; multiplicative groups; additive groups; logarithm operator; powers of arithmetic functions; multiplicative functions

Full Text: DOI