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A commutative algebra for oriented matroids. (English) Zbl 1016.52014

Discrete Comput. Geom. 27, No. 1, 73-84 (2002).

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^\ell$, with linear defining forms $\{\phi_1, \dots, \phi_n\}$. In his work on cohomology of local systems, *K. Aomoto* introduced the algebra $AO(\mathcal{A})$ of rational forms generated by $(\phi_{i_1} \cdots \phi_{i_k})^{-1}$ where $\{i_1, \dots, i_k\}$ ranges over all independent subsets of $\mathcal{A}$ [Sugaku Expo. 9, 99-116 (1996; Zbl 0787.33001)]. *P. Orlik* and *H. Terao* in [Nagoya Math. J. 134, 65-73 (1994; Zbl 0801.05019)] constructed a presentation of the commutative algebra $AO(\mathcal{A})$ similar to the more familiar (skew-commutative) Orlik-Solomon algebra $OS(\mathcal{A})$. A crucial difference is that the presentation of $AO(\mathcal{A})$ depends explicitly on the coefficients of dependence relations among the forms ϕ_i. By contrast, $OS(\mathcal{A})$ is defined in terms of the underlying matroid of $\mathcal{A}$. Orlik and Terao ask whether in fact $AO(\mathcal{A})$ is determined by the underlying matroid. The main point of the paper under review is to show that this is not the case.

This result is accomplished by studying a closely related algebra $\mathbb{A}(\mathcal{M})$, a combinatorial analogue of $AO(\mathcal{A})$ defined in terms of the oriented matroid $\mathcal{M}$ associated with $\mathcal{A}$. The algebras $\mathbb{A}(\mathcal{M})$ and $AO(\mathcal{A})$ are isomorphic in case there is a spanning set of dependence relations among the ϕ_i having all coefficients equal to ± 1 or 0. It is shown that the first examples of *C. Eschenbrenner* and *M. Falk* [J. Algebr. Comb. 10, 189-199 (1999; Zbl 0955.52010)], which have different underlying matroids, nevertheless have isomorphic $\mathbb{A}$ algebras, as is the case also for their Orlik-Solomon algebras. These examples satisfy the condition stated above, and thus provide a negative answer to the question of Orlik and Terao.

The author also establishes a "no-broken-circuit" basis result for the oriented matroid algebra $\mathbb{A}(\mathcal{M})$. An interesting notion of "combinatorial basis" is introduced: this is a basis $\{b_1, \dots, b_n\}$ of $\mathbb{A}^1(\mathcal{M})$ for which a product of distinct elements $b_{i_1} \cdots b_{i_k} = 0$ if and only if $\{i_1, \dots, i_k\}$ is dependent in the underlying matroid. The same definition makes sense for $AO(\mathcal{A})$ and $OS(\mathcal{A})$. The algebra $\mathbb{A}(\mathcal{M})$ is said to fix the underlying matroid if the only combinatorial bases are obtained by permutation of the standard basis. It is shown that, in case the algebra $\mathbb{A}(\mathcal{M})$ fixes the underlying matroid, $\mathbb{A}(\mathcal{M})$ actually determines the oriented matroid $\mathcal{M}$, and thus the homotopy type of the complexified complement M.

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