Little, David P.

Combinatorial aspects of the Lascoux-Schützenberger tree. (English) Zbl 1018.05102

Adv. Math. 174, No. 2, 236-253 (2003).

For a permutation σ of the symmetric group S_n, let $\operatorname{Red}(\sigma)$ be the set of all reduced decompositions $\sigma = s_{a_1} \cdots s_{a_l}$, i.e. presentations of σ as a product of minimal length with respect to the generators $s_i = (i, i+1)$, $i = 1, \dots, n-1$. As an approach to the fundamental problem to determine the cardinality of $\operatorname{Red}(\sigma)$ for a fixed σ, *R. P. Stanley* [Eur. J. Comb. 5, 359-372 (1984; Zbl 0587.20002)] introduced a function $F_\sigma(X)$. He showed that $F_\sigma(X)$ is symmetric and for the permutation $\sigma = (n, n-1, \dots, 1)$, the element of longest length, the number of reduced words is equal to the number of standard Young tableaux of staircase shape $(n-1, n-2, \dots, 1)$. Stanley also conjectured that the symmetric function $F_\sigma(X)$ is Schur positive. The conjecture was confirmed by *P. Edelman* and *C. Green* [in: Combinatorics and algebra, Proc. Conf., Boulder/Colo. 1983, Contemp. Math. 34, 155-162 (1984; Zbl 0562.05008) and Adv. Math. 63, 42-99 (1987; Zbl 0616.05005)] using the technique of balanced tableaux.

The main contribution of the paper under review is the construction of a correspondence Θ_σ which sends the reduced decomposition $w \in \operatorname{Red}(\sigma)$ to a pair $(\alpha(w), T(w))$, where $\alpha(w)$ is a Grassmanian permutation (a permutation with only one descent) and $T(w)$ is a standard tableau of shape $\lambda'(\alpha(w))$. The main idea is to associate a line diagram to each word w which illustrates the trajectories of the numbers $1, 2, \dots, n$ as they are rearranged by successive simple transpositions. The proof that Θ_σ is a bijection is quite simple and its properties can be established in a straightforward manner. This gives an elementary proof of the Schur positivity of the Stanley symmetric functions. The author also obtains a simple and purely combinatorial proof of the version of the Littlewood-Richardson rule given by *A. Lascoux* and *M.-P. Schützenberger* [Lett. Math. Phys. 10, 111-124 (1985; Zbl 0586.20007)].

Reviewer: Vesselin Drensky (Sofia)

MSC:

05E05 Symmetric functions and generalizations
05E15 Combinatorial aspects of groups and algebras (MSC2010)

Keywords:

Stanley symmetric functions; Lascoux-Schützenberger tree; reduced factorizations of symmetric group; Littlewood-Richardson rule

Full Text: DOI

References:

[1] P. Edelman, C. Greene, Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the weak Bruhat order of \textit{S}n, in: Combinatorics and Algebra (Boulder, CO, 1983), Amer. Math. Soc., Providence, RI, 1984, pp. 155-162.

[2] Edelman, P.; Greene, C., Balanced tableaux, Adv. in math., 63, 1, 42-99, (1987) · Zbl 0616.05005

[3] A.M. Garsia, The saga of reduced factorizations of elements of the symmetric group, Publications du LaCIM, Université du Québec á Montréal, Canada, Vol. 29, 2002.

[4] Lascoux, A.; Schützenberger, M.-P., Schubert polynomials and the Littlewood-Richardson rule, Lett. math. phys., 10, 2-3, 111-124, (1985) · Zbl 0586.20007

[5] Stanley, R.P., On the number of reduced decompositions of elements of Coxeter groups, European J. combin., 5, 4, 359-372, (1984) · Zbl 0587.20002