Villani, Cédric

Topics in optimal transportation. (English) Zbl 1106.90001

Graduate Studies in Mathematics 58. Providence, RI: American Mathematical Society (AMS) (ISBN 0-8218-3312-X/hbk). xvi, 370 p. (2003).

This is a very interesting book: it is the first comprehensive introduction to the theory of mass transportation with its many – and sometimes unexpected – applications. In a novel approach to the subject, the book both surveys the topic and includes a chapter of problems, making it a particularly useful graduate textbook.

In 1781, Gaspard Monge defined the problem of "optimal transportation" (or the transferring of mass with the least possible amount of work), with applications to engineering in mind. In 1942, Leonid Kantorovich applied the newborn machinery of linear programming to Monge's problem, with applications to economics in mind. In 1987, Yann Brenier used optimal transportation to prove a new projection theorem on the set of measure preserving maps, with applications to fluid mechanics in mind.

Each of these contributions marked the beginning of a whole mathematical theory, with many unexpected ramifications. Nowadays, the Monge-Kantorovich problem is used and studied by researchers from extremely diverse horizons, including probability theory, functional analysis, isoperimetry, partial differential equations, and even meteorology.

These notes are definitely not intended to be exhaustive, and should rather be seen as an introduction to the subject. Their reading can be complemented by some of the reference texts which have appeared recently. In particular, mention should be made of the two-volume work by *S. T. Rachev* and *L. Rüschendorf* [Mass transportation problems. Vols. I, II. Springer, New York (1998; Zbl 0990.60500)], which depicts many applications of Monge-Kantorovich distances to various problems, together with the classical theory of the optimal transportation problem in a very abstract setting; the survey of Evans, which can be considered as an introduction to the subject and describes several applications of the L^1 theory (i.e., when the cost function is a distance) which are not covered in these notes; the extremely clear lecture notes by *L. Ambrosio* [Lect. Notes Math. 1812, 1–52 (2003; Zbl 1047.35001)], centred on the L^1 theory from the point of view of calculus of variations; and also the lecture notes by *J. Urbas* ["Mass transfer problems", Lecture notes, Univ. Bonn, 1997–1998; per bibl.], which are a marvelous reference for the regularity theory of the Monge-Ampère equation arising in mass transportation. Also recommended is a very pedagogical and rather complete article recently written by *L. Ambrosio* and *A. Pratelli* [Lect. Notes Math. 1813, 123–160 (2003; Zbl 1065.49026)], focused on the L^1 theory, from which many remarks and examples are extracted here.

The present volume does not go too deeply into some of the aspects which are very well treated in the above-mentioned references: in particular, the L^1 theory is just sketched, and so is the regularity theory developed by Caffarelli and by Urbas. Several topics are hardly mentioned, or not at all: the application of mass transportation to the problem of shape optimization, as developed by Bouchitté and Buttazzo; the fascinating semi-geostrophic system in meteorology, whose links with optimal transportation are now understood thanks to the amazing work of Cullen, Purser and collaborators; or applications to image processing, developed by Tannenbaum and his group. On the other hand, this text is a good elementary reference source for such topics as displacement interpolation and its applications to functional inequalities with a geometrical content, or the differential viewpoint of Otto, which has proven so successful in various contexts (like the study of rates of equilibration for certain dissipative equations). The proofs are kept as simple as possible throughout the book, are understandable by non-expert students. Many results without proofs, either to convey a better intuition, or to give an account of recent research in the field.

Originating from a graduate course, the present volume is intended for graduate students and researchers, covering both theory and applications. Readers are only assumed to be familiar with the basics of measure theory and functional analysis. (Mainly cited from the publisher notes)

Reviewer: Olaf Ninnemann (Berlin)

MSC:

- 90-02 Research exposition (monographs, survey articles) pertaining to operations research and mathematical programming
- 28D05 Measure-preserving transformations
- 35B65 Smoothness and regularity of solutions to PDEs
- 35J60 Nonlinear elliptic equations
- 49N90 Applications of optimal control and differential games
- 49Q20 Variational problems in a geometric measure-theoretic setting
- 90B20 Traffic problems in operations research