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Non-Kähler compact complex manifolds associated to number fields. (English) Zbl 1071.32017

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The authors consider a class of compact complex manifolds X which they construct as quotients of $\mathbb{H}^s \times \mathbb{C}^t$ by properly discontinuous group actions, $s,t > 0$. These manifolds have the property that $b_1(X) = s \leq \dim H^1(X, \mathcal{O}_X)$, hence they are not Kähler. Moreover

$$H^0(X, \Omega^1_X) = H^0(X, K_X^{\otimes k}) = \{0\}$$

for all $k \geq 1$. The quotients $X_{s,1}$ of $\mathbb{H}^s \times \mathbb{C}$ admit locally conformal Kähler metrics. In particular, the example $X_{2,1}$ with Betti numbers $b_1 = b_5 = 2$, $b_3 = 0$, $b_{2i} = 1$, $0 \leq i \leq 3$, answers the question whether compact complex manifolds with locally conformal Kähler structure and $b_{2i+1} \in 2\mathbb{N}, i \geq 0$, are necessarily Kähler.

The construction is based on well known facts from geometric number theory: Let $\mathcal{O}_K$ be the ring of integers of the algebraic number field K and $\mathcal{O}_K^*$ the group of units in $\mathcal{O}_K$. Assume that K admits s embeddings $\rho_1, \dots, \rho_s$ of K into $\mathbb{R}$ and $2t$ non-real embeddings $\sigma_1, \overline{\sigma}_1, \dots, \sigma_t, \overline{\sigma}_t$ into $\mathbb{C}$. $\mathcal{O}_K$ can be realized as a lattice of rank $s + 2t = [K : \mathbb{Q}]$ in $\mathbb{C}^s \times \mathbb{C}^t$ via the injection $\tau : K \to \mathbb{C}^{s+t}, \tau(a) := (\rho_1(a), \dots, \rho_s(a), \sigma_1(a), \dots, \sigma_t(a))$, operating on $\mathbb{C}^s \times \mathbb{C}^t$ by translations and leaving $\mathbb{H}^s \times \mathbb{C}^t$ invariant. The quotient $(\mathbb{H}^s \times \mathbb{C}^t)/\tau(\mathcal{O}_K)$ is diffeomorph to $(\mathbb{R}_{>0})^s \times (S^1)^{s+2t}$. The image of the logarithmic representation $\lambda : \mathcal{O}_K^* \to \mathbb{R}^{s+t}$, $\lambda(u) := (\log|\rho_1(u)|, \dots, \log|\rho_s(u)|, \log|\sigma_1(u)|^2, \dots, \log|\sigma_t(u)|^2)$, is a lattice of maximal rank in the linear hyperplane $\{(x_1, \dots, x_{s+t}) \in \mathbb{R}^{s+t} \mid \sum_{i=1}^{s+t} x_i = 0\}$ and $\{(\log|\rho_1(u)|, \dots, \log|\rho_s(u)|) \mid u \in U\}$ is a lattice of rank s in $\mathbb{R}^s$ for suitable subgroups U of $\mathcal{O}_K^*$. Combining these representations yields a properly discontinuous action of the semidirect product $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$, and the quotient $X = X(K,U)$ is diffeomorph to a fiber bundle over $(S^1)^s$ with $(S^1)^{s+2t}$ as fiber.

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MSC:

32J18	Compact complex n-folds
32M17	Automorphism groups of $\mathbb{C}^n$ and affine manifolds
11H56	Automorphism groups of lattices

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