Rivoal, Tanguy

Linear independence of values of polylogarithms. (Indépendance linéaire des valeurs des polylogarithmes.) (French) Zbl 1079.11038

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The polylogarithms are $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$. Let $a \geq 2$ be an integer and $\alpha = p/q$ be a rational with $0 < |\alpha| < 1$. Let $\delta_\alpha(a) = \dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\mathrm{Li}_1(\alpha) + \cdots + \mathbb{Q}\mathrm{Li}_a(\alpha))$. For every $\varepsilon > 0$, there is a constant $A = A(\varepsilon, p, q)$ such that if $a \geq A \geq 1$ then $\delta_\alpha(a) \geq \frac{1-\varepsilon}{1+\log 2} \log a$. So the $\mathrm{Li}_s(\alpha)$ with $s = 1, 2, \ldots$ contain infinitely many $\mathbb{Q}$-linearly independent numbers (and infinitely many irrationals). The proof rests on properties of the nearly-poised hypergeometric functions

$$N_{n,a,r}(z) = n!^{a-r} \sum_{k=1}^{\infty} \frac{(k-1)(k-2)\cdots(k-rn)}{k^a(k+1)^a \cdots (k+n)^a} z^{-k}$$

and Nesterenko's criterion for linear independence. Since $\mathrm{Li}_s(1) = \zeta(s)$, this is an interesting complement to Rivoal's remarkable theorem that infinitely many $\zeta(2n+1)$ are irrational.

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MSC:

11J72	Irrationality; linear independence over a field
11M41	Other Dirichlet series and zeta functions
33B15	Gamma, beta and polygamma functions

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Polylogarithms; hypergeometric functions; Nesterenko's criterion

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