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Geodesic geometry and fixed point theory. II. (English) Zbl 1083.53061

García-Falset, Jesús (ed.) et al., Proceedings of the international conference on fixed-point theory and its applications, Valencia, Spain, July 13–19, 2003. Yokohama: Yokohama Publishers (ISBN 4-946552-13-8/hbk). 113-142 (2004).

The paper under review is the second part of the author's earlier paper [Geodesic geometry and fixed point theory. Girela Alvarez, Daniel (ed.) et al., Seminar of mathematical analysis. Proceedings of the seminar which was held at the Universities of Malaga and Seville, Spain, September 2002-February 2003. Sevilla: Universidad de Sevilla, Secretariado de Publicaciones. 195–225 (2003; Zbl 1058.53061)]. It deals with fixed point theory in metric spaces, in particular in Aleksandrov Re_K domains (for Re_K domains, see the foundational papers by *A. D. Aleksandrov* [Tr. Mat. Inst. Steklova 38, 5–23 (1951; Zbl 0049.39501) and "Über eine Verallgemeinerung der Riemannschen Geometrie", Ber. Riemann-Tagung Forsch.-Inst. Math. 33–84 (1957; Zbl 0077.35702)]), also known as $CAT(K)$ spaces.

The paper starts with a short review of Aleksandrov spaces (in his paper, the author refers to Aleksandrov Re_K domains as "the so-called $CAT(K)$ spaces of M. Gromov") and the author's previous results. The results presented in the paper deal with fixed point theorems in Re_0 domains, approximate fixed point theorems and fixed point theorems for asymptotically nonexpansive mappings. The author starts with the following general result: if K is a non-empty bounded closed convex subset of a complete Re_0 domain and $f : K \to K$ is a nonexpansive mapping, then the fixed point set of f is nonempty, closed and convex. Next, the author extends one of his previous theorems to *locally nonexpansive* mappings by proving that for such mappings from a connected bounded open set D in a complete Re_0 domain to the domain itself, which can be extended continuously to $\overline{D}$, the following alternative holds: either f has a fixed point in D or $\inf_{X\in\partial D}\{d(x, f(x))\} \leq \inf_{X\in D}\{d(x, f(x))\}$.

Among results related to the fixed point theorems, the author also proves that a nonexpansive mapping of the product of a metric space which has the fixed point property for nonexpansive mappings and a complete bounded Re_0 domain, relative to the metric d_∞ (max-metric) on the product, has at least one fixed point. If the second space is an $\mathbb{R}$-tree, the boundedness condition can be relaxed. A subset S of a metric space $(\mathcal{M}, d)$ *has the approximate fixed point property for nonexpansive mappings* if $\inf_{x\in S}(d(x, T(x))) = 0$, for every nonexpansive mapping $T : S \to S$. It is proved that the product of a metric space with the *approximate fixed point property for nonexpansive mappings* and a bounded convex subset of a possibly incomplete Re_0 domain has the approximate fixed point property for nonexpansive mappings. A mapping $T : M \to M$ is called *asymptotically nonexpansive* if there is a sequence $\{k_n\}_{n=1,2,\dots}$ of real numbers with $k_n \to 1$ such that $d(T^n(x), T^n(y)) \leq k_n d(x, y)$, for every $x, y \in M$. The author proves that if K is a bounded closed and convex subset of a complete Re_0 domain, then every asymptotically nonexpansive mapping $T : K \to K$ has a fixed point.

For the entire collection see [Zbl 1063.47054].

Reviewer: I. G. Nikolaev (Urbana)

MSC:

53C45	Global surface theory (convex surfaces à la A. D. Aleksandrov)
58C30	Fixed-point theorems on manifolds
05C75	Structural characterization of families of graphs
47H09	Contraction-type mappings, nonexpansive mappings, A-proper mappings, etc.
54H25	Fixed-point and coincidence theorems (topological aspects)
51K10	Synthetic differential geometry
05C05	Trees

Keywords:

nonexpansive mappings; fixed points; CAT(0) spaces; $\mathbb{R}$-trees