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Uniqueness of meromorphic functions sharing three values. (English) Zbl 1085.30027

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For a meromorphic function f in the complex plane, let $T(r,f)$ denote the Nevanlinna characteristic, and let $S(r,f)$ be any quantity that satisfies $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly on a set of finite linear measure. For $a\in\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ and a positive integer k or $k=\infty$, let $N(r,a;f|\le k)$ denote the counting function of the a-points (poles if $a=\infty$) of f with multiplicity at most k. Then the usual counting function is given by $N(r,a;f)=N(r,a;f|\le\infty)$, where every a-point is counted according to its multiplicity. If every a-point is counted only once, the corresponding function is denoted by $\overline{N}(r,a;f)$. Furthermore, let $E_k(a;f)$ be the set of a-points of f, where an a-point of multiplicity m is counted m times if $m\le k$ and $k+1$ times if $m>k$. It is said that two meromorphic functions f and g share a value $a\in\widehat{\mathbb{C}}$ with weight k, or shortly they share (a,k), if $E_k(a;f)=E_k(a;g)$. Obviously, f and g share a IM (ignoring multiplicities) or CM (counting multiplicities) if and only if f and g share $(a,0)$ or (a,∞), respectively.

A theorem of *H. Ueda* [Kodai Math. J. 6, 26–36 (1983; Zbl 0518.30029)] states that if two non-constant meromorphic functions share the values 0, 1 and ∞ CM, and if

$$\limsup_{r\to\infty}\frac{N(r,0;f)+N(r,\infty;f)}{T(r,f)}<\frac{1}{2}, \tag{$*$}$$

then either $f\equiv g$ of $fg\equiv 1$. The special case for entire functions of finite order was already done by *M. Ozawa* [J. Anal. Math. 30, 411–420 (1976; Zbl 0337.30020)]. An improvement was achieved by *H. X. Yi* [Kodai Math. J. 13, 363–372 (1990; Zbl 0712.30029)] by replacing $(*)$ by the weaker condition

$$\limsup_{r\to\infty}\frac{N(r,0;f|\le 1)+N(r,\infty;f|\le 1)}{T(r,f)}<\frac{1}{2}. \tag{$**$}$$

W. R. Lü and *H. X. Yi* replaced the bound $\frac{1}{2}$ in $(**)$ by 1 and obtained that then

$$f\equiv\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}\quad\text{and}\quad g\equiv\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

where s and k are relatively prime positive integers with $1\le s\le k$ and γ is a non-constant entire function. The functions $f=(e^\gamma-1)^2$ and $g=e^\gamma-1$ show that this result is not true in general if f and g share the value 0 only IM. In this paper, the authors consider the question whether it is possible to relax the nature of sharing the value 0. Their result reads as follows.

Theorem. Let f and g be two distinct non-constant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1)>(m+1)^2$. If $(**)$ holds, then f and g satisfy the relations

$$\left(1+\frac{\alpha}{f}-\frac{1}{f}\right)^s\equiv\alpha^{s+t}\quad\text{and}\quad\left(1+\frac{1}{g\alpha}-\frac{1}{g}\right)^s\equiv\alpha^{-(s+t)},$$

where α is a non-constant meromorphic function such that $\overline{N}(r,0;\alpha)+\overline{N}(r,\infty;\alpha)=S(r,f)$, and s, t are relatively prime non-zero integers with $s>0$ and $s+t\ne 0$. In particular, the above result of W. R. Lü and H. X. Yi remains valid if f and g share $(0,1)$, $(1,\infty)$ and (∞,∞).

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