**Allendoerfer, Carl B.; Weil, André**
**The Gauss-Bonnet theorem for Riemannian polyhedra.** (English) Zbl 0060.38102
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The classical Gauss-Bonnet formula expresses the curvatura integral of a curved polygon on a surface in $E_3$ in terms of angles of the polygon and the integral of geodesic curvatures of its edges. This formula gives rise when applied to closed orientable surfaces in $E_3$ to a remarkable relation between a differential invariant and a topological invariant (Euler characteristic) of the surface. So it has been conjectured since many years ago that there might be hidden important relationship between differential invariants and topological invariants of Riemannian manifolds and Gauss-Bonnet formula will be a key for it.

About 1926, H. Hopf generalized this formula to 1) even dimensional closed Riemannian manifolds $M_n$ which can be imbedded in $E_{n+1}$ and 2) closed space forms, i. e. complete closed Riemannian manifolds of constant curvature. Both of these manifolds are very special among all Riemannian manifolds. Hopf proposed to generalize the formula for general Riemannian manifolds. The problem is to find a suitable differential form of $n$-th degree such that its integral over the whole manifold is equal to the Euler Poincaré characteristic of the manifold.

In 1940, *W. Fenchel* [J. Lond. Math. Soc. 15, 15–22 (1940; Zbl 0026.26401)] and *C. B. Allendoerfer* [Am. J. Math. 62, 243–248 (1940; Zbl 0024.35101)] independently solved this problem for the case of even dimensional closed Riemannian manifolds which can be imbedded in $E_{n+q}$ $(q \geq 1)$. But Allendoerfer-Weil's paper was the first which solved the problem for general closed Riemannian manifolds. Their method is as follows:

First take a cell $\sigma^n$ of sufficiently fine subdivision of the given $n$-dimensional Riemannian manifold. Imbed the $\sigma^n$ in Euclidean space $E_N$ $(N = \frac{n(n+1)}{2})$. This is possible by Schläfli-Cartan's immersion theorem. We construct then the tube $\sigma^n(\dot{r}$ in $E_N$, i. e. the set of all points $y$ such, that the distance from $y$ to $\sigma^n$ is at most $r$ for sufficiently small $r$. We denote the point of $\sigma^n$ nearest to $y$ by $z(y)$ and denote the vector from $z(y)$ to $y$ by $w(y)$. The mapping $y \to w(y)$ is a continuous map of the tube on the $N$ dimensional disc $\Sigma_N$ with degree (Abbildungsgrad) +1, so $\int dw^1\, dw^2 \cdots dw^N$ extended over the tube is equal to the volume of the disc. This equation is nothing but the generalization of the Gauss-Bonnet formula for a polygon on a surface in $E_3$. Practically the $n$-cell $\sigma^n$ in $E_N$ has many edges (faces) of different dimensions, so the explicit calculation of the formula requires extensive analysis of outer angles of $n$-cells and the theory of tubes developed by H. Weyl. The Gauss-Bonnet formula for closed Riemannian manifold follows immediately from that of $n$-cells.

*S. S. Chern* could get a very simple and intuitive proof of the Gauss-Bonnet theorem as the simplest example of a more general theory of fibre bundles. We shall explain his main idea in the case of closed orientable Riemannian manifold. Chern introduces the tangent sphere bundle $F^{(1)}$ of $M_n$. It is a $(2n-1)$-dimensional manifold. Chern proves first that the differential form $\Omega$ which appears in the Gauss-Bonnet formula $\int_{M_n} \Omega = \chi(M_n)$ is a derived form (i. e. $\Omega = d\Pi$) if we regard it as a form in $F^{(1)}$. Construct a continuous field of unit vectors over $M_n$ with isolated singular points and denote its image in $F^{(1)}$ by $V_n$. Then $\int_{M_n} \Omega = \int_{V_n} \Omega = \int_{\partial V_n} \Pi$ (by Stokes theorem). However, $\partial V_n$ corresponds exactly to the singular points of the vector field defined in $M_n$, the sum of their indices is by a well known theorem of H. Hopf equal to $\chi(M_n)$. It can be easily seen that if we take a differential form $\Omega$ which belongs to the characteristic cohomology class of $M_n$, then $\int \Omega = c\chi(M_n)$ ($c$: const).

*S. S. Chern* has proved that $\Omega$ we had considered above belongs to the characteristic class. In this way his method connects directly to the problem of expressing characteristic classes of fibre bundles defined over differentiable manifolds in terms of integrals of some differential invariants. Thus he could express Whitney's invariant for a normal vector field of a submanifold $M_n$ in a Riemannian manifold $M_{2n}$, in terms of an integral of differential geometric invariant and in general discussed the problem to express the characteristic classes of sphere bundles over $M_n$ by integral formulas of some differential invariants of the manifold.

*A. Lichnerowicz* generalized the Gauss-Bonnet theorem to a certain class of Finsler manifolds, i. e. to manifolds of Berwald-Cartan, which are characterized by the vanishing of a kind of curvature tensors.

Reviewer: S. Sasaki

For a scan of this review see the web version.

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