Busemann, H.

The geometry of geodesics. (English) Zbl 0112.37002

Pure and Applied Mathematics, 6. New York: Academic Press, Inc. x, 422 p. (1955).

In the thirteen years since the publication of the author's "Metric methods in Finsler spaces and in the foundations of geometry" [Princeton, NJ: Princeton University Press (1942; Zbl 0063.00672)] enough work has been done on metric geometry to fill several other monographs. However, the present volume aims at the broader purpose of presenting a very readable and almost self-contained treatment of most of the field. The basic concepts have been refined, some proofs made more elegant, and, most important, the methods applied to so many problems that it is now evident how fruitful is this "geometric approach to qualitative problems in intrinsic differential geometry".

Chapter I, "The basic concepts", is concerned with elementary metric-space topology. Rectifiable curves are discussed, and Menger's convexity condition is introduced: for any two points x and z, there is a point y with $xy + yz = xz$ (xy denoting the distance between x and y.) Two more axioms are needed before the existence of geodesics with the usual properties can be asserted: an axiom of local prolongability of segments, and a condition guaranteeing uniqueness of prolongation. G-spaces are finitely compact metric spaces which satisfy these axioms and have the convexity property. The remainder of the chapter develops the important basic properties of G-spaces. (Most of this material is in [Zbl 0063.00672], although the discussion of multiplicity of a geodesic is new.) Two-dimensional G-spaces are shown to be manifolds, and those which are straight (where geodesics are unique shortest connections) are characterized.

Chapter II treats Desarguesian G-spaces, i.e., G-spaces which may be imbedded in projective space so that geodesics become straight lines. To obtain straight planes with this property, a form of Desargues' Theorem is postulated; for higher dimensions the condition that any three points lie in a plane (a two-dimensional G-space, relative to the original metric) is sufficient. A Desarguesian Riemannian G-space is either euclidean, hyperbolic, or elliptic. Because of their relation to Finsler spaces, the most important non-Riemannian Desarguesian spaces are the Minkowskian. A more elegant approach than appeared in [Zbl 0063.00672] is given to Minkowskian geometry. The chapter ends with a discussion of another non-Riemannian case: the Hilbert geometries constructed in convex sets of affine space.

The basic tool in a treatment of perpendiculars for a straight G-space is the notion of convex spheres; for parallels, the idea of asymptote is a natural generalization. Chapter III, entitled "Perpendiculars and parallels", uses these concepts to give characterizations of elliptic geometry (dimension greater than two), spherical spaces, Minkowski spaces, and Euclidean spaces. Most of this appeared in [Zbl 0063.00672], but the present arrangement is much more readable. Finally, the relations are given which hold between the assumptions made in this chapter and the conjugate or focal points of the calculus of variations.

In geometry the theory of covering spaces finds some of its most elegant applications, for the covering transformations are rigid motions. Chapter IV develops this theory for G-spaces, obtaining most of the classical results on the fundamental group, fundamental domain, and on the existence of closed geodesics. Spaces whose universal coverings are straight generalize the interesting Finsler case in which there are no conjugate points. All such G-space metrizations of a two-dimensional torus are characterized. (There are many, although, by E. Hopf's Theorem, the only Riemannian one is flat.) The analogue of Poincaré's unit-circle model of hyperbolic geometry is used to establish the existence of transitive geodesics on compact G-surfaces of higher genus with straight covering spaces, provided there are no "parallels" and the geodesics have a divergence property.

Just as in Riemannian geometry, the local condition which most easily implies that the universal covering space is straight is that of non-positive curvature. Chapter V investigates "The influence of the sign of the curvature on the geodesics," using the definition of non-positive curvature which requires that the line joining the midpoints of two sides of a small geodesic triangle be at most half as long as the third side. The term "space with convex capsules" is used for the (demonstrably) weaker definition of non-positive curvature introduced by *F. P. Pedersen* [Mat. Tidsskr. B 1952, 66–89 (1952; Zbl 0048.40502)], where the capsules (loci equidistant from segments) are locally convex. Most of the known results on the behavior of geodesics in Riemannian spaces of negative curvature can be carried over, often using only the weaker hypothesis. The second half of this chapter attacks the problem of curvature of surfaces from a different quarter, by introducing measures of angles. The results, which are almost all contained in a paper of the author's [Can. J. Math. 1, 279–296 (1949; Zbl 0034.10201)], show that many consequences of the Gauss-Bonnet theorem do not depend on differentiability, let alone the Riemannian character of the geometry.

From the point of view of the foundations of geometry, the sixth and last chapter is the most important. In short, the result left partially solved in [Zbl 0063.00672] is now given a definitive form: If every point of a G-space has flat local bisectors, the universal covering space is euclidean, hyperbolic, or spherical. By means of this theorem, various characterizations of these elementary spaces are given which are answers to the Helmholtz-Lie problem. Finally, Wang's theorem is proved, which characterizes every compact (and odd-dimensional non-compact) space, convex in the sense of Menger, having a pairwise transitive group of motions. (The even-dimensional, non-compact case has been announced by J. Tits.) Indicative of the care taken to make the book as self-contained as possible is the section reviewing the hermitian, quaternionic, and Cayley elliptic geometries.

The remarkable thing about this book, covering as it does so many topics of geometry in the large, is the amount of material which represents original research of the author's. The presentation is very clear; motivation is given both for the theorems and for some of the involved proofs. The list of problems, solved and unsolved, will be a stimulation to many. A real service has been done for the current revival of interest in geometry and geometrical methods.

Reviewer: Leon W. Green (Minneapolis) (MR0075623)

For a scan of this review see the web version.

MSC:

53-02	Research exposition (monographs, survey articles) pertaining to differential geometry
52-02	Research exposition (monographs, survey articles) pertaining to convex and discrete geometry
53Cxx	Global differential geometry
52Axx	General convexity

Keywords:
metric geometry; convex geometry; integral geometry