Bosch, S.; **Güntzer, Ulrich**; **Remmert, Reinhold**

Non-Archimedean analysis. A systematic approach to rigid analytic geometry. (English)

Zbl 0539.14017

Grundlehren der Mathematischen Wissenschaften, 261. Berlin etc.: Springer Verlag. XII, 436 p. DM 168.00 (1984).

Since the introduction of rigid analytic spaces by Tate in 1961, rigid analytic geometry (i.e. the analog of complex analytic geometry over complete non archimedean fields) has had a growing importance in algebraic geometry, number theory, p-adic analysis, etc. However, the basic results of the theory had remained scattered in various papers, sometimes difficult to find, and some fundamental notions have undergone various changes as the theory was making progress. Thus, the subject was not so easily accessible to non-specialists, and there was an increasing need for a systematic account of the theory of rigid analytic spaces in addition to the introductory book of *J. Fresnel* and *M. van der Put* ["Géométrie analytique rigide et applications", Prog. Math. 18 (1981; Zbl 0479.14015)]. The present book should meet that need.

The book is divided in three parts. The first one, "Linear ultrametric analysis and valuation theory", is devoted to prerequisites and fundamentals in ultrametric analysis and analytic function theory. In chapter 1, the basic notions concerning ultrametric norms on abelian groups and rings are introduced. They cover such topics as the reduction functor, power multiplicative semi-norms and smoothing procedures, strictly convergent power series with coefficients in a semi-normed ring. Chapter 2 is devoted to the theory of normed modules and vector spaces. The bulk of the chapter is a discussion of various types of normed vector spaces characterized by the existence of (eventually topological) basis for which the norm of the projections of a vector are closely related to the norm of the vector itself. Of particular interest for rigid analytic geometry is the case of normed vector spaces of countable type; for these spaces, the chapter includes a proof of the "lifting theorem", which gives conditions under which an algebraic basis of the reduction of a normed space can be lifted as a topological orthonormal basis of the space. The third and last chapter of part A discusses the extensions of norms and valuations. In addition to the case of field extensions and classical valuation theory, the chapter introduces for K-algebras over a non-archimedean field K the notions of spectral norm and supremum norm; a special attention is given to the case of Banach algebras, and to the behaviour of the supremum norm under finite extensions, which will be of great importance in the theory of affinoid algebras (e.g. for the proof of the "maximum modulus principle").

The second part, "Affinoid algebras", develops the algebraic foundations of rigid analytic geometry. Chapter 4 is devoted to the study of the Tate algebra $T_n = K\langle X_1, \dots, X_n\rangle$ of strictly convergent power series in n variables over a complete non archimedean field K. A particularly important result is the Weierstrass division and preparation theorem, from which are derived some basic properties: T_n is noetherian, factorial, normal, its ideals are closed, etc. Chapter 5 then proceeds with the theory of affinoid algebras, i.e. quotient algebras of algebras T_n, which play the same role in rigid analytic geometry as the affine algebras in algebraic geometry. After establishing the analog of Noether's normalisation theorem, which is one of the most useful tools in the study of affinoid algebras, some classical properties of the supremum semi-norm, such as the maximum modulus principle, are proved. The chapter ends with properties of the sub-algebra Å of elements of semi-norm ≤ 1 in an affinoid algebra, including the important finiteness theorem for the reduction functor.

The third part of the book deals with the theory of rigid analytic spaces. Chapter 7 develops the local theory. To any affinoid algebra A is attached its maximal spectrum SpA, and the basic problem in order to get a good notion of analytic space is to define a topology on SpA having reasonable properties from the point of view of analytic continuation. This is achieved thanks to the construction of a Grothendieck topology on SpA. A subset U of SpA is called an affinoid subdomain if there exists an affinoid A-algebra A', such that any homomorphism of affinoid algebras $A \to B$ for which SpB is sent in U factors uniquely through A'. For example, the rational domains, defined by inequalities $|f_i(x)| \leq |f_0(x)|$, $i = 1, \dots, r$, where $f_0, f_1, \dots, f_r$ are elements of A generating the unit ideal, are affinoid subdomains, and are shown to have good stability properties. A theorem of Gerritzen and Grauert, proved at the end of chapter 7, implies in particular that any affinoid subdomain is a finite union of rational domains. Affinoid subdomains and finite coverings by affinoid subdomains can then be used as generators for a Grothendieck topology, as explained in the beginning of chapter 9. Thanks to Tate's acyclicity theorem, to which is devoted chapter 8, it is possible to define the sheaf of analytic functions on SpA as a sheaf for that topology, associating to any affinoid subdomain the corresponding algebra. - Grothendieck topologies allow then to glue these local models to develop the global theory of rigid analytic spaces. This is done in the last sections of chapter 9, which cover the theory of coherent modules, finite morphisms, closed analytic subvarities, separated and proper morphisms. Kiehl's finiteness theorem for direct images is discussed, although the proof is not included. The book ends with the example which was the starting point of the theory: Tate's uniformization for elliptic curves with bad reduction.

Reviewer: P.Berthelot

MSC:

14G20	Local ground fields in algebraic geometry
32P05	Non-Archimedean analysis (should also be assigned at least one other classification number from Section 32-XX describing the type of problem)
12J10	Valued fields
32-02	Research exposition (monographs, survey articles) pertaining to several complex variables and analytic spaces
14-02	Research exposition (monographs, survey articles) pertaining to algebraic geometry
32B05	Analytic algebras and generalizations, preparation theorems
46S10	Functional analysis over fields other than $\mathbb{R}$ or $\mathbb{C}$ or the quaternions; non-Archimedean functional analysis

Keywords:
rigid analytic geometry; rigid analytic spaces; ultrametric analysis; affinoid algebras; Tate algebra; coherent modules; finiteness theorem for direct images; uniformization for elliptic curves with bad reduction