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**The meromorphic continuation of a zeta function of Weil and Igusa type.** (English)

Zbl 0571.12009
Invent. Math. 85, 493-514 (1986).

Let $K_1$ denote a finite algebraic extension of $\mathbb{Q}_p$. Let $K_d$ denote the unique unramified extension of $K_1$ of degree $d$. For $d \geq 1$ let $R_d$ denote the ring of integers of $K_d$ and $P_d$ its unique maximal ideal. Let $q = \operatorname{card} R_1/P_1$. Then $R_d/P_d \simeq \mathbb{F}_{q^d}$, the finite field with $q^d$ elements. Let $f(x) = f(x_1, \dots, x_n)$ denote a polynomial in $n$ variables with coefficients in $R_1$. Let

$$N_{e,d} = \operatorname{card}\{x \pmod{P_d^e)^{(n)}} \mid f(x) \equiv 0 \pmod{P_d^e}$$

and define $P(w,z)$ by $P(w,z) = \sum_{d\geq 1}\sum_{e\geq 0} N_{e,d} q^{-nde} w^d z^e$. The above series is a holomorphic function on the product of the two open unit discs in $\mathbb{C}^2$. The coefficient of $z$ is $Q_1(w) = \sum_{d\geq 1} N_{1,d} q^{-nd} w^d$, which is rational as a consequence of the rationality of the Weil zeta function. The coefficient of $w^d$ is $P_d(z) = \sum_{e\geq 0} N_{e,d} q^{-nde} z^e$ which is the Igusa zeta function for the field $K_d$ and hence is rational.

In this paper we show that $P(w,z)$ has a meromorphic continuation to $\mathbb{C}^2$. This is a best possible result since $P(w,z)$ is not in general rational. We completely characterize those polynomials for which it is rational.

In the process of proving the above result we also prove two other results. The first is that a zeta function which is analogous to the Weil zeta function, namely $Z_e(T) = \exp \sum_{d\geq 1} N_{e,d} T^d/d$ for any $e > 1$, is rational. The second is that the Igusa zeta function is an invariant of $f(x)$ which only depends upon the degree $d$ of the extension in a simple way.

Reviewer: Diane Meuser

**MSC:**

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|---|---|
| 11S40 | Zeta functions and $L$-functions |
| 11S05 | Polynomials |
| 32A20 | Meromorphic functions of several complex variables |
| 14G10 | Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture) |



**Keywords:**

polynomial in several variables; Weil zeta function; Igusa zeta function; meromorphic continuation; rationality of zeta functions

**Full Text:** DOI EuDML

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