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User's guide to spectral sequences. (English) Zbl 0577.55001

Mathematics Lecture Series, 12. Wilmington, Delaware: Publish or Perish, Inc. XIII, 423 p. $ 40.00 (1985).

Used originally in the computation of the homology of fibrations (as the Leray-Serre spectral sequence), nowadays spectral sequences have become an indispensable tool in concrete and abstract computations of homology groups. The apparatus of spectral sequences is used not only in algebraic topology but also in algebra, algebraic geometry and algebraic K-theory. The present book is designated as a "user's manual for the class of complicated algebraic gadgets known as spectral sequences". The book is written both for beginners and experts, topologists and algebraists. It includes sufficient details in exposition and examples, enough facts about the inner workings of the theory of spectral sequences and plenty of information for experts.

The book is divided into three parts totaling eleven chapters. The first three chapters form Part I called "Algebra". This part contains, in detail, the algebra of spectral sequences and furthermore, how this formalism can be applied to a topological problem. Chapter 1 is a "gentle" introduction to the technique of the first quadrant spectral sequences. An important number of examples is discussed. Chapter 2 treats more of the structural features of spectral sequences including the settings in which they arise and replacing the bigraded vector spaces of chapter 1 by $\mathbb{Z}$-bigraded modules over a commutative ring with unity. In chapter 3, comparison theorems are proved and examples of two spectral sequences that are useful in homological algebra are given. Exercises on the topics in chapters 1, 2 and 3 follow this chapter.

Part II is called "Topology" and is comprised of chapters 4 through 9. This part covers the most important problems of the book treating the three classical examples of spectral sequences that are found in homotopy theory: the Leray-Serre spectral sequence, the Eilenberg-Moore spectral sequence and the Adams spectral sequence. In chapter 4 the categories of CW-complexes and of simplicial sets are presented. Both, a CW-complex and a simplicial set give rise to a filtered space and in this way the algebraic techniques developed in Part I may be applied to the computation of the homotopy invariants of such spaces. In this chapter most proofs are omitted and are replaced by references.

Chapter 5 and 6 treat the Leray-Serre spectral sequence. In chapter 5 the motivating ideas for the construction of the spectral sequence are given, the E^1-term and the differential d^1 are identified, the cohomology version of the spectral sequence and its relevant multiplicative structure are considered, the classical theorems of Leray and Hirsch and of Borel and Serre are derived, the cohomology of various Lie groups is computed and some of the implications in homotopy theory of possible computations on the loop space by the spectral sequence are explored. Two highlights of chapter 5 are the generalization of a theorem of Marston Morse on the existence of infinitely many distinct geodesics joining two points in a Riemannian manifold and the proof that $\pi_i(S^{2n+1})$ is finite for $i > 2n+1$. The details of the proof of the homology version of the Leray-Serre spectral sequence are postponed to the appendices in the last paragraph of the chapter.

Chapter 6 contains deeper basic properties of the Leray-Serre spectral sequence and also some applications. Thus one proves the isomorphism $\pi_4(S^2) \cong \mathbb{Z}/2\mathbb{Z}$, the Kudo transgression theorem, the theorems of Cartan and Serre and one studies the cohomology of classifying spaces of fibre bundles and the problem of computing the characteristic classes associated to a bundle. In the last paragraph of chapter 6 some different settings which lead to the construction of a spectral sequence for fibrations, as the result of Fadell and Hurewicz and the Dress construction, are discussed. This section is "not for the novice"! (beside a few other subsections of the book). Exercises on the topics in chapters 5 and 6 follow.

Chapters 7 and 8 treat the Eilenberg-Moore spectral sequence. Chapter 7 refers to differential homological algebra, the Eilenberg and Moore theorems, examples, and a computational tool. In chapter 8 one considers more substantial applications of the Eilenberg-Moore spectral sequences. Thus, one studies the cohomology of homogeneous spaces and the differentials in the Eilenberg-Moore spectral sequence and one considers the Eilenberg-Moore spectral sequence as a Künneth theorem. A lot of exercises for chapters 7 and 8 are given. Chapter 9 is dedicated to the classical Adams spectral sequence "as constructed in the days before spectra". This chapter includes some motivating ideas, the spectral sequence, computations, two deep theorems of Adams about $\mathrm{Ext}_{A_2}^{s,t}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, the May spectral sequence and some of the techniques that lead to the determination of differentials in the Adams spectral sequence. Exercises follow.

Part III is called "Sins of omission" and is comprised of chapters 10 and 11. Chapter 10 contains some spectral sequences which have arisen outside of topology (in algebra, algebraic geometry and algebraic K-theory): the Lyndon-Hochschild-Serre spectral sequence, spectral sequences for rings and modules, the Grothendieck spectral sequence, the Leray spectral sequence, the Brown-Gersten spectral sequence and Tomason's descent spectral sequence. In chapter 11 one continues the catalogue of spectral sequences and includes some of the further applications of these objects in algebraic and differential topology. By their birth the examples of this chapter may be classified as follows: those of homological origin (Bockstein), spectral sequences associated to mappings (Cartan-Leray and Federer), those involving spectra (Atiyah-Hirzebruch, the universal coefficient spectral sequence and Miller spectral sequence) and the Adams-type spectral sequences (D. W. Kahn, Massey-Peterson, Curtis-Rector-MIT). The final paragraph contains two examples that are not in the mainstream of homotopy theory. The first deals with Lie algebras and their cohomology (the Van Est spectral sequence) and the second applies to the study of manifolds and Poincaré duality (the dihomology spectral sequence).

The bibliography and an index of notions finish the book. The author's convention for bibliographic references is somewhat complicated but by this one allows the pursuit of a problem in its advancement.

In the reviewer's opinion this is an excellent book, containing a wealth of interesting material, masterly written.

Reviewer: Ioan Pop (Iaşi)

MSC:

55-02	Research exposition (monographs, survey articles) pertaining to algebraic topology
55Txx	Spectral sequences in algebraic topology
57T35	Applications of Eilenberg-Moore spectral sequences
18G40	Spectral sequences, hypercohomology
57T15	Homology and cohomology of homogeneous spaces of Lie groups
55R40	Homology of classifying spaces and characteristic classes in algebraic topology
55R20	Spectral sequences and homology of fiber spaces in algebraic topology
55Q40	Homotopy groups of spheres
55Q45	Stable homotopy of spheres
55M05	Duality in algebraic topology
57R19	Algebraic topology on manifolds and differential topology
55U10	Simplicial sets and complexes in algebraic topology
17B56	Cohomology of Lie (super)algebras
20J05	Homological methods in group theory
18F25	Algebraic K-theory and L-theory (category-theoretic aspects)
18Gxx	Homological algebra in category theory, derived categories and functors
18G15	Ext and Tor, generalizations, Künneth formula (category-theoretic aspects)
22E41	Continuous cohomology of Lie groups

Keywords:

homotopy groups of spheres; cohomology of Lie groups; Federer; spectral sequence; Atiyah-Hirzebruch spectral sequence; cohomology of Lie algebras; Bockstein spectral sequence; first quadrant spectral sequences; bigraded modules; comparison theorems; CW-complex; simplicial set; filtered space; loop space; geodesics; Leray-Serre spectral sequence; classifying spaces of fibre bundles; characteristic classes; Eilenberg-Moore spectral sequence; differential homological algebra; Adams spectral sequence; May spectral sequence; Lyndon-Hochschild-Serre spectral sequence; Grothendieck spectral sequence; Leray spectral sequence; Brown-Gersten spectral sequence; descent spectral sequence; universal coefficient spectral sequence; Miller spectral sequence; Van Est spectral sequence; Poincaré duality; dihomology spectral sequence