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Clifford theory for group-graded rings. (English) Zbl 0583.16001

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The best way to treat Clifford's theory about the effect of a normal subgroup N on the irreducible representations of a finite group H is to consider the group algebra FH of H over any field F as a ring R naturally graded by the factor group $G = H/N$. Then the 1_G-component R_1 of R is the group algebra FN of N, while the σ-component R_σ, for any $\sigma \in G$, is the linear span $F\sigma$ of the elements of the coset $\sigma \in H/N$. In particular, R_σ contains at least one unit $u \in \sigma$ of R for each $\sigma \in G$, i.e. R is a crossed product of G over R_1.

It is well-known that Clifford's theory extends almost without modification to simple modules over arbitrary crossed products R of the finite group G over rings R_1 (it doesn't work when G is infinite, even for group algebras). The subject of the present paper is the surprising observation that all parts of his theory can be made to work for simple modules over completely arbitrary rings R graded by a finite group G, provided one is willing to redefine (in a natural way) the notion of "induction" of an R_I-module L to an R-module L^R whenever I is a subgroup of G and $R_I = \sum_{\sigma\in I} R_\sigma$ is the corresponding I-graded subring of R. One must also redefine "G-conjugacy" for simple R_1-modules. Then our results can be stated as:

(1) The restriction M_{R_1} to R_1 of any simple R_1- module M is isomorphic to the direct sum $m \times (U_1 \oplus \cdots \oplus U_n)$ of m copies of $U_1 \oplus \cdots \oplus U_n$, where $m > 0$ and $U_1, \ldots, U_n$ are representatives for the distinct isomorphism classes in some "G-conjugacy class" of simple R_1- modules. - (2) If M "lies over" $U = U_1$ as in (1), then the U-primary component $M\{ U\}$ of the semi-simple R_1-module M_{R_1} is a simple $R_{G\{U\}}$-module lying over U, where $G\{ U\}$ is the stabilizing subgroup in G of the isomorphism class of the simple R_1-module U. Furthermore, M is isomorphic to the R-module "induced" from $M\{ U\}$. Indeed, if L is any simple $R_{G\{U\}}$-module lying over U, then the "induced" R-module L^R is simple, lies over U, and has a U-primary component $L^R\{U\}$ isomorphic to L. - (3) The $R_{G\{U\}}$-endomorphism ring E of the $R_{G\{U\}}$-module S "induced" from the R_1-module U is naturally a crossed product of $G\{ U\}$ over the division ring $E_1 \simeq End_{R_1}(U)$. In the case of right modules, any simple $R_{G(U)}$-module L lying over U yields a simple E-module $L < U >= Hom_{R_G\{U\}}(S, L)$ such that L is isomorphic to the $R_{G\{U\}}$-module $L < U > \otimes_E S$. Conversely, any simple E-module K determines a simple $R_{G\{U\}}$-module $K \otimes_E S$ lying over U such that $K \simeq (K \otimes_E S) < U >$. Similar statements can be made for left modules.

Of course, the maps of modules in (2) and (3) can be extended to additive functors forming equivalences of suitable categories. Evidently the combination of (1), (2) and (3) reduces the study of simple modules over rings R graded by a finite group G to that of simple modules over crossed products E of subgroups $G\{ U\}$ of G over endomorphism rings E_1 of simple R_1-modules U.

MSC:

16W50	Graded rings and modules (associative rings and algebras)
16S34	Group rings
20C15	Ordinary representations and characters
16W20	Automorphisms and endomorphisms
16D30	Infinite-dimensional simple rings (except as in 16Kxx)
16S50	Endomorphism rings; matrix rings
20C05	Group rings of finite groups and their modules (group-theoretic aspects)

Keywords:

irreducible representations; group algebra; Clifford's theory; simple modules; crossed products; restriction; endomorphism rings

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