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Linear forms in logarithms on commutative algebraic groups. (Formes linéaires de logarithmes sur les groupes algébriques commutatifs.) (French) Zbl 0651.10023

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This article contains some very precise lower bounds for linear forms in logarithms with respect to an arbitrary commutative group variety. They generalize the classical results for conventional logarithms due to A. Baker and others.

Suppose G is defined over a number field K and is of dimension $d+1$. Select basis elements over K of the tangent space to G at its origin, and write exp for the resulting map from $\mathbb{C}^{d+1}$ to $G(\mathbb{C})$. Let $L(z)=\beta_0 z_0+\ldots+\beta_d z_d$ be a non-zero linear form on $\mathbb{C}^{d+1}$ with coefficients in K, and denote by W the subspace of $\mathbb{C}^{d+1}$ defined by $L(z)=0$. Let v be in $\mathbb{C}^{d+1}$ such that $\exp(v)$ is in $G(K)$. Suppose finally that for every connected algebraic subgroup G' of G whose tangent space lies in W this tangent space does not contain v. Then the Analytic Subgroup Theorem of *G. Wüstholz* (see for example, [Prog. Math. 31, 329–336 (1983; Zbl 0534.10026)]) shows that $L(v)\neq 0$.

The present article gives a positive lower bound for $|L(v)|$ in these circumstances. In order to include the classical results on ordinary logarithms, the group G is assumed to have the form $G=G_0\times G_1\times\cdots\times G_k$, where G_0 is the additive group and $G_1=\cdots=G_{d_1}$ is the multiplicative group for some d_1 with $0\leq d_1\leq k$. Write δ_i for the dimension of G_i $(0\leq i\leq k)$, and define $\rho_i=0$ $(i=0)$, $\rho_i=1$ $(1\leq i\leq d_1)$ and $\rho_i=2$ otherwise. Break the tangent space into corresponding factors, select basis elements still defined over K, and let $\exp_i$ be the map on the corresponding factor of $\mathbb{C}^{d+1}$ $(0\leq i\leq k)$. Choose norms $\|\cdot\|$ on these latter factors. Also assume each G_i embedded in projective space $(0\leq i\leq k)$. Then the main result of the authors is the existence of a constant c, depending only on the quantities introduced in the present paragraph, with the following property.

Suppose the point v above can be written as $v=(1,v_1,\ldots,v_k)$ with $v_i\neq 0$ and $\gamma_i=\exp_i(v_i)$ in $G_i(K)$ $(1\leq i\leq k)$. Write h for the absolute logarithmic Weil height on projective space over K. Let $D=[K:\mathbb{Q}]$, and let $B,E,V_1,\ldots,V_k$ be real numbers satisfying $\log V_i\geq\max\{h(\gamma_i),\|v_i\|^{\rho_i}/D\}$, $B\geq D\log V_i$ $(1\leq i\leq k)$ and $\log B\geq h(\beta_j)$ $(0\leq j\leq d)$ as well as $E\leq\min\{BD, e(D\log V_i)^{1/\rho_i}/\|v_i\|\}$ $(1\leq i\leq k)$. Then

$$\log|L(v)|>-cD^{d+d_2+2}(\log B)^{d_2+1}\log(DE)(\log E)^{-d-d_2-1}V,$$

with $V=\prod_{i=1}^{k}(\log V_i)^{\delta_i}$. For purposes of comparison, note that if $d_2=0$ (the classical situation of ordinary logarithms) this estimate compares very well with the sharpest results to date [see for example, *M. Waldschmidt*, Acta Arith. 37, 257–283 (1980; Zbl 0357.10017)]. But if $d_2>0$ the results are essentially completely new, except for certain cases involving complex multiplication.

The method of proof is rather ingenious. Of course it relies on the full power of multiplicity estimates, as established first by *G. Wüstholz* [Habilitationsschrift, Wuppertal 1982] and sharpened by *P. Philippon* [(*): Bull. Soc. Math. Fr. 114, 355–383 (1986; Zbl 0617.14001)]. But the new idea is an inductive "descent" that succeeds only because the estimates of (*) are nearly best possible. Another ingredient of the proof is an interesting inequality of *D. Bertrand* and *P. Philippon* [Ill. J. Math. 32, 263–280 (1988; Zbl 0618.14020)].

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14A05 Relevant commutative algebra

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