Maddy, Penelope
Believing the axioms. I. (English) Zbl 0652.03033

The author wishes to examine the foundations of set theory from the beginning, without accepting any sacred cows (such as the Zermelo-Fraenkel axioms). She attempts a careful examination of underlying principles and rules of thumb that guide set theorists in choosing axioms and in deciding what is to be considered as good or bad evidence for adopting additional assumptions. However, she admits to being influenced strongly by the members of the Cabal seminar, especially in connection with determinacy and large cardinals.

The first section is a relatively brief study of the Zermelo-Fraenkel axioms. She points out that those axioms were inspired by Zermelo’s desire to justify his proof of the well-ordering principle, not by a desire to avoid the paradoxes. There are interesting discussions of the origins and various justifications of each of the axioms. Here and throughout the paper, the author picks out broad and vague general principles that set theorists sometimes appeal to when they are supporting or attacking specific propositions; examples are “limitations of size”, the “iterative conception”, “one step back from disaster”, “realism”, and “cantorian finitism”. The only reservation I have here concerns the axiom of infinity. The author sees it (correctly, in my opinion) as “the bold and revolutionary hypothesis that launched modern mathematics”. She describes Cantor’s breakthrough in terms of “having the audacity to assume that [collections of elements] could be infinite”. It may be more accurate that Cantor’s boldness was in treating infinite collections as mathematical objects in their own right. The ultimate effect of this section is to convince the reader of the claim that “from the beginning the process of adopting set-theoretic axioms has not been a simple matter of noting down the obvious”.

The second and major section of this paper, on the continuum problem, is quite interesting. The history of the problem is outlined, including some of the consistency and independence results. Then the author summarizes a host of arguments against and in favor of the continuum hypothesis. The job is done very well. There are also excellent discussions about the truth (or, more often, the falsity) of the axiom of constructibility. The third section of the paper concerns small large cardinals and is primarily devoted to the axiom of inaccessibles. The fourth section has to do with measurable cardinals. Here, there is much interesting material on contemporary results; the author tries to make a strong case in favor of the existence of measurable cardinals. A sequel to this paper will take up large large cardinals and determinacy.

Reviewer: E. Mendelson

MSC:
03E30 Axiomatics of classical set theory and its fragments
00A30 Philosophy of mathematics
03A05 Philosophical and critical aspects of logic and foundations

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