Anderson, David F.; **Bouvier, Alain**; **Dobbs, David E.**; **Fontana, Marco**; **Kabbaj, Salah**
On Jaffard domains. (English) Zbl 0657.13011
Expo. Math. 6, No. 2, 145-175 (1988).

Let R be a commutative ring with identity and let $X_1, X_2, ..., X_r$ be algebraically independent indeterminates over R. Let dim(R) denote the Krull dimension of R. Then the following inequalities are known to hold:

$$r + \dim(R) \leq \dim(R[X_1, X_2, ..., X_r]) \leq r + (r+1)\dim(R).$$

However, *P. Jaffard* [Mém. Sci. Math. 146 (1960; Zbl 0096.025)] has shown that if R is a domain, with quotient field $K \neq R$, and L is an algebraic extension of R, then, for any positive integer n the following conditions are equivalent: (i) R has a (L-)valuation overring of dimension n but no (L-)valuation overring of dimension greater than n, (ii) R has a (L-)overring of dimension n but no (L-)overring of dimension greater than n, (iii) $\dim(R[X_1, X_2, ..., X_n]) = 2n$, (iv) for all $r \geq n-1$, $\dim(R[X_1, X_2, ..., X_r]) = r + n$.

Because of this, the present authors say that a domain R, not a field, is Jaffard if $\dim(R)$ is finite and $\dim(R[X_1, X_2, ..., X_r]) = r + \dim(R)$ for each $r \geq 0$. Examples of Jaffard domains include finite-dimensional domains which are either (a) Noetherian, (b) Prüfer, (c) universally catenarian, or (d) stably strong S-domains. Here a domain R is universally catenarian if every finitely generated R-algebra A is catenary, while strong S-domains were first considered by *S. Malik* and *J. L. Mott* [J. Pure Appl. Algebra 28, 249-264 (1983; Zbl 0536.13001)]. The authors note in their introduction that domains satisfying the altitude inequality formula, with examples (a)-(d) as special cases, are also Jaffard. Here R satisfies the altitude inequality formula if, for each finite extension S of R and each P in Spec(S), we have

$$ht(P) + tr.\deg_{K(P\cap R)}(K(P)) \leq ht(P\cap R) + tr.\deg_R(S),$$

as discussed on, for example, page 119 of the book by *H. Matsumura* ["Commutative ring theory", Cambr. Studies Adv. Math. 8 (1986; Zbl 0603.13001)]. - However an example is given of a Jaffard domain not satisfying the inequality formula.

The first section of the paper discusses transfer properties of Jaffard domains. For example, it is proved that integral extensions preserve and reflect the Jaffard property and that the localization of a Jaffard domain need not be Jaffard. The latter motivates an investigation of locally Jaffard domains, in particular of when these are Jaffard. Equivalent conditions are given for a monoid domain, over an abelian torsion-free monoid, to be Jaffard.

Section 2 of the paper looks at necessary and sufficient conditions for certain pullback constructions to be Jaffard, with the $D+M$ construction getting special attention. - The final section has eight interesting examples which illustrate that several of the results in the first two sections are best possible.

Reviewer: J.Clark

MSC:

13C15	Dimension theory, depth, related commutative rings (catenary, etc.)
13F20	Polynomial rings and ideals; rings of integer-valued polynomials
13G05	Integral domains

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polynomial extension; Jaffard domains; integral extensions; localization; pullback