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Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. (English) Zbl 0661.53027

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Let M be an n-dimensional Riemannian manifold with tangent space T_pM at the point $p \in M$. An element $z \in T_pM \otimes \mathbb{C}$, the complexified tangent space, is said to be isotropic if $(z, z) = 0$. A complex linear subspace $V \subset T_pM \otimes \mathbb{C}$, is totally isotropic if $z \in V$ then $(z, z) = 0$. By definition, the curvature of a Riemannian manifold M is positive on totally isotropic two-planes if the complex sectional curvature $K(\sigma) > 0$ whenever $\sigma \subseteq T_pM \otimes \mathbb{C}$ is a totally isotropic two-plane situated at any point $p \in M$. The authors prove the following main theorem: Let M be a compact simply connected n-dimensional Riemannian manifold which has positive curvature on totally isotropic two-planes where $n \geq 4$. Then M is homeomorphic to a sphere. This theorem was announced in [the second author, Bull. Am. Math. Soc., New Ser. 14, 279-282 (1986; Zbl 0589.53048)].

Let $\mathcal{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM$ be the curvature operator at p. $\mathcal{R}$ is said to be (k,1)-positive if $\{\sum_{i=1}^k < \mathcal{R}(\omega_i), \omega_i >\} > 0$ for any orthonormal set $\{\omega_1, ..., \omega_k\}$ of k elements from $\Lambda^2 T_pM$, each of which has rank ≤ 1. The following corollary is proved: Let M be a compact simply connected n-dimensional Riemannian manifold, where $n \geq 2$. If the curvature of M satisfies one of the following conditions, then M is homeomorphic to a sphere: (i) M has positive curvature operator, (ii) M has (2,2)-positive curvature operators, or (iii) M has strictly pointwise (1/4)-pinched sectional curvatures. Also a theorem regarding the structure of stable minimal two- spheres in Riemannian manifolds with nonnegative curvature is proved.

As a corollary it is proved that an odd-dimensional compact Riemannian manifold whose sectional curvatures are nonstrictly pointwise (1/4)- pinched must have vanishing second homotopy group. Finally it is remarked that if M satisfies all the hypotheses of the main theorem except for being simply connected, it is not necessarily true that the universal cover of M is a sphere, but still $\pi_k(M) = 0$ for $2 \leq k \leq n/2$ can be concluded.

Reviewer: C.S.Houh

MSC:

53C20	Global Riemannian geometry, including pinching	Cited in **9** Reviews
53C42	Differential geometry of immersions (minimal, prescribed curvature, tight, etc.)	Cited in **86** Documents

Keywords:

sphere theorem; complexified tangent space; positive curvature; totally isotropic two-planes; curvature operator; pinched sectional curvatures; minimal two-spheres; second homotopy group

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