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Hypergraphs do not jump. (English) Zbl 0663.05047

Combinatorica 4, 149-159 (1984).

Let $G = G(V, E)$ be a graph on n vertices with V = vertex set and E = edge set $\subset V \times V$). The ratio of the number edges in the graph to the total number possible is called the density of G, i.e. $d(G) = |E|/\binom{n}{2}$. There is an unexpected "jump" in the density of a subgraph versus the graph itself. It follows by a theorem of Erdős, Stone and Simonovitis that for any positive integer $m \geq 2$, real $0 \leq \alpha \leq 1$, and n sufficiently large, any graph on n vertices having density greater than α contains a subgraph on m vertices having density greater than $\alpha + c$, where c is some fixed, positive constant not depending on m or n. For example, in the class of complete ℓ-partite graphs whose partition classes are of size k there exist subgraphs which are complete m-points with densities $= 1$ that exceed $d(G)$ by more than $c = 1/(\ell + 1)$ for arbitrary $k > \ell$ and $2 \leq m \leq \ell$ (since in this case $d(G) = (k\ell + k)/(k\ell - 1)$).

In this interesting paper, the authors extend the problem to include r-uniform hypergraphs – graphs whose "edges" are r-element subsets of V (in this more general setting, $d(G) = |E|/\binom{n}{r}$. The precise definition for jump used here is: a real number $0 \leq \alpha \leq 1$ is a jump for r provided that for any positive c and any integer $m \geq r$, an r-uniform hypergraph with $n > n_0(\varepsilon, m)$ vertices and density at least $\alpha + \varepsilon$ contains a subgraph on m vertices with density at least $\alpha + c$, where $c = c(\alpha)$ does not depend on ε and m. By use of the Lagrange function on graphs and an analysis of complete ℓ-partite r-uniform graphs, the authors prove that the numbers $1 - 1/\ell^{r-1}$ for $\ell = 2r + 1, 2r + 2, \ldots$ are not jumps if $r \geq 3$. This settles a question of Erdős who has offered a \$ 1,000 prize for the answer.

Reviewer: D. Kay

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References:

[1] P. Erdös, On extremal problems of graphs and generalized graphs,Israel J. Math. 2 (1965), 183–190. · Zbl 0129.39905 · doi:10.1007/BF02759942

[2] P. Erdös, Problems and Results on Graph and Hypergraphs, Similarities and Differences. · Zbl 0725.05051

[3] P. Erdös andM. Simonovits, A limit theorem in graph theory,Studia Sci. Mat. Hung. Acad. 1 (1966), 51–57.

[4] P. Erdös andA. H. Stone, On the structure of linear graphs,Bull. Amer. Math. Soc. 52 (1946), 1087–1091. · Zbl 0063.01277 · doi:10.1090/S0002-9904-1946-08715-7

[5] V. Jarník,Differentialrechnung, Prague 1956.

[6] G. Katona, T. Nemetz andM. Simonovits, On a graph-problem of Turán,Mat. Lapok 15 (1964), 228–238.

[7] P. Turán, On an extremal problem in graph theory (in Hungarian),Mat. Fiz. Lapok 48 (1941), 436–452.