Let $X$ be a Riemann surface. $X$ is hyperbolic if it is covered by the unit disk and $X$ is of finite type if it is obtained from a compact surface by deleting a finite number of points. Let $Q(X)$ be the space of holomorphic quadratic differentials $\phi(z)dz^2$ on $X$, such that

$$\|\phi\| = \iint_X |\phi(z)|dxdy < \infty.$$ 

Let $B_X$ denote the open unit ball in the Banach space $Q(X)$. Now suppose $f: Y \to X$ is a covering map. Then there is a natural operator $\Theta: Q(Y) \to Q(X)$ defined by pulling back over the branches of $f^{-1}$ and summing over all of the branches. This is the classical Poincaré theta operator when $f$ is the universal covering of the unit disk onto a Riemann surface $X$. As an operator between the Banach spaces $Q(Y)$ and $Q(X)$, the norm of $\Theta$ is less than or equal to 1. The chief result of this paper describes the types of coverings $f$ for which the norm of $\Theta$ is strictly less than 1. Say a covering $f$ is amenable if there are large balls with small boundary in a graphic caricature of the covering. If $X$ is hyperbolic and of finite type, then its universal cover is nonamenable.

**Theorem.** Let $Y \to X$ be a covering of a hyperbolic Riemann surface $X$. Either:
1. The covering is amenable, and $\Theta(B_Y) = B_Y$, or
2. The covering is nonamenable, and the closure of $\Theta(B_Y)$ is contained in the interior of $B_X$.

As a corollary, if $G$ is a nonabelian Fuchsian group acting on the unit disk $\Delta$ such that $X = \Delta/G$ is a compact Riemann surface of finite type, then the norm of $\Theta$ is less than 1 and the inclusion mapping from the Teichmüller space of $X$ into universal Teichmüller space is contracting. One of the elements of the proof involves analyzing the flex of certain points on the boundary of the unit ball in the Banach space $Q(Y)$. The notion of flex is a sort of local version of the idea of reflexivity of Banach spaces. Suppose $\Theta(\psi) = \phi$. To prevent loss of mass, the phase of $\psi$ must nearly agree with that of the pull back of $\phi$ to $Y$, at least over a region $Y_0$ which contains most of the mass of $|\psi|$. From the lemma on flex, one shows that agreement of phase implies the mass distribution of $\psi$ mimics that of $\phi$, which in the large is determined by the combinatorics of the covering $Y \to X$. For a nonamenable covering, most of the mass of $Y_0$ will be near its boundary, where the pairing is inefficient by a definite amount; this forces $\|\Theta\| < 1$.

The arguments are carried further to estimate the dependence of $\|\Theta\|$ on moduli.

**Theorem.** Let $X$ be a hyperbolic Riemann surface of finite type, $Y \to X$ an infinite-sheeted covering space with finitely generated fundamental group. Then $\|\Theta_{Y/X}\| < c(n, L) < 1$, where $c(n,L)$ is a function of $n$ the number of generators of $\pi_1(Y)$ and $L$ the length of the shortest geodesic on $X$ and $c(n,L)$ depends continuously on $L$. By using this result coupled with the theory of geometric limits of quadratic differentials, the author gives a new, analytic proof of the existence of a fixed point for Thurston’s “skinning” map. The skinning map is a key tool in Thurston’s construction of hyperbolic structures on 3-manifolds.

Reviewer: F.P.Gardiner

**MSC:**

- 30C70 Extremal problems for conformal and quasiconformal mappings, variational methods
- 30F35 Fuchsian groups and automorphic functions (aspects of compact Riemann surfaces and uniformization)
- 30C62 Quasiconformal mappings in the complex plane
- 11F12 Automorphic forms, one variable
Keywords: expansion constant; hyperbolic 3-manifolds; quadratic differentials; Poincaré theta operator; amenable; Teichmüller space; flex

Full Text: DOI EnDML

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