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On the homology of classifying spaces for foliated products. (English) Zbl 0674.57023

Foliations, Proc. Symp., Tokyo 1983, Adv. Stud. Pure Math. 5, 37-120 (1985).

[For the entire collection see Zbl 0627.00017.]

Classifying spaces for Γ-structures, now called Haefliger structures, were introduced by *A. Haefliger* [Topology 9, 183-194 (1970; Zbl 0196.269)] to give a homotopy classification of foliations on open manifolds up to integrable homotopy. Thurston in his fundamental work on foliations extended Haefliger's result to obtain a homotopy classification on all manifolds up to concordance [*W. Thurston*, Comment. Math. Helv. 49, 214-231 (1974; Zbl 0295.57013) and Ann. Math., II. Ser. 104, 249-269 (1976; Zbl 0347.57014)]. As is the case with many classification results in differential topology the work is simply shifted into homotopy theory, in the case of foliations the computation of the homotopy groups of the classifying space $B\overline{\Gamma}_n^r$ of C^r-Haefliger structures of codimension n with a (homotopy class of) trivialization of its normal bundle. Haefliger [loc. cit.] showed that $\pi_i(B\overline{\Gamma}_n^r) = 0$ for $i \leq n$. Translated via obstruction theory into geometry this amounted into obtaining the trivial fact that all 1-dimensional subbundles of the tangent bundle of a manifold is homotopic to an integrable one. The first non-trivial result that $\pi_{n+1}(B\overline{\Gamma}_n^r) = 0$ if $r \neq n+1$, was substantially more difficult to prove. It consists of two difficult parts, first the construction of a homology equivalence $B\overline{Diff_c\mathbb{R}^n} \to \Omega^n B\bar{\Gamma}_n^r$ which is valid for all n and r, and then the calculation that $H_1(B\overline{Diff}_c^r\mathbb{R}^n) = 0$ if $r \neq n+1$. The first result is due to J. Mather in codimension 1 and to Thurston in general. The second result is due to Mather. Here $B\overline{Diff}_c^r\mathbb{R}^n$ is the homotopy fibre of the map $BDiff_c^{r,\delta}\mathbb{R}^n \to BDiff_c^r\mathbb{R}^n$, where $Diff_c^r\mathbb{R}^n$ is the group of C^r-diffeomorphisms of $\mathbb{R}^n$ with compact support and with the standard topology, while $Diff_c^{r,\delta}$ is the corresponding discrete group.

The publication of the results of Mather and Thurston date back to 1974- 1976, and although it is almost a conjecture that for $r \geq 2$ the space $B\overline{\Gamma}_n^r$ is 2n-connected, not much progress was made until the work of the author. The paper under review should be considered a major stepping stone in deciding whether $\pi_{n+i}B\overline{\Gamma}_n^r = 0$ or equivalently $H_i(B\overline{Diff}_c^r) = 0$ for $1 < i \leq n$. He adapts to the C^r-situation the original approach of *J. N. Mather* [Topology 10, 297-298 (1971; Zbl 0207.219)] to the calculation of $H_i(B\overline{Homeo_c}\mathbb{R}^n)$. The idea might seem simple but the actual implementation of it is quite ingenious and highly non- trivial. His main result $H_2(B\overline{Diff}_c^r\mathbb{R}^n) = 0$ if $1 \leq r < [n/2]$, and $H_i(B\overline{Diff}_c^r\mathbb{R}^n) = 0$ if $1 \leq r < [(n+1)/i] - 1$, $i \geq 1$, raises the obvious question whether the connectivity of $B\overline{\Gamma}_n^r$, $r \geq 2$, does depend on r (it is known that $B\overline{\Gamma}_n^r$ is contractible for $r = 0, 1$).

Reviewer: E.Vogt

MSC:

57R32 Classifying spaces for foliations; Gelfand-Fuks cohomology
55R35 Classifying spaces of groups and H-spaces in algebraic topology
57R50 Differential topological aspects of diffeomorphisms

Keywords:

homotopy groups of the classifying space of C^r-Haefliger structures; homology of the classifying space of the group of diffeomorphisms of n-space with compact support with the discrete topology; group of C^r-diffeomorphisms of $\mathbb{R}^n$ with compact support