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Wulff theorem and best constant in Sobolev inequality. (English) Zbl 0676.46031

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Let $f:\mathbb{R}^2\to\mathbb{R}$ be a positively homogeneous function of degree one, lower semi-continuous, with $f(x)>0$ if $x\neq 0$. For each such function one defines a convex set of $\mathbb{R}^2$, W_f,

$$W_f=\{x^*\in\mathbb{R}^2:\quad f^*(x^*)\leq 0\}=\{x^*\in\mathbb{R}^2:\quad f^0(x^*)\leq 1\},$$

where f^*, resp. f^0, is the Legendre transform, resp. the polar transform, of f. Let $(u,v)\in W^{1,1}_{per}(a,b)\times W^{1,1}_{per}(a,b)$ with $u'^2+v'^2\neq 0$ a.e. in (a,b). Let

$$F(u,v)=\int_a^b f(v'(\theta),-u'(\theta))d\theta,\quad m(u,v)=\int_a^b (v'(\theta)u(\theta)-u'(\theta)v(\theta))d\theta.$$

Then the following inequality holds

$$(*)\quad F^2(u,v)-4|W_f|m(u,v)\geq 0,$$

where $|W_f|$ is the Lebesgue measure of W_f. Equality holds if and only if (u,v) is a parametrization of ∂W_f.

Inequality (*) is a generalized isoperimetric inequality. Indeed, if f is the Euclidean norm and (u,v) a parametric representation of the boundary ∂A of a region A, then $F(u,v)=\ell(\partial A)$ and $m(u,v)=|A|$, where $\ell(\partial A)$ is the length of ∂A and $|A|$ the area of A. In that case W_f is the unit Euclidean disk and $|W_f|=\pi$. The proof of (*) is a consequence of a generalized Wirtinger inequality

$$\inf\{\int_{-1}^1(f(v',-u'))^2d\theta/\int_{-1}^1(f^0(u,v))^2d\theta:\quad(u,v)\in\mathcal{M}\}=(|W_f|)^2,$$

where

$$\mathcal{M}=\{(u,v)\in H^1(-1,1)\times H^1(-1,1);\quad u(-1)=u(1),$$

$$v(-1)=v(1),\quad\int_{-1}^1 f^0(u,v)\partial f^0/\partial u(u,v)d\theta=\int_{-1}^1 f^0(u,v)\partial f^0/\partial v(u,v)d\theta=0\}.$$

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MSC:

46E35 Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems

49J27 Existence theories for problems in abstract spaces

46E39 Sobolev (and similar kinds of) spaces of functions of discrete variables

Keywords:

Legendre transform; polar transform; generalized isoperimetric inequality; generalized Wirtinger inequality