Kiang, Tsai-han

The theory of fixed point classes. Transl. of the rev. 2nd Chinese ed. (English) Zbl 0676.55001

Berlin etc.: Springer-Verlag; Beijing: Science Press. xi, 174 p. (1989).

In Nielsen fixed point theory, the fixed points of a continuous self- mapping f of a finite polyhedron are assigned to equivalence classes, the fixed point classes of the title. Certain of the classes are identified as "essential" using algebraic topology, and the Nielsen number N(f) of the map is defind to be the number of essential classes. Every map homotopic to f has at least N(f) fixed points.

The author was introduced to this subject by Solomon Lefschetz not long after the publication of the paper of *J. Nielsen*'s that established the subject [Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen. I, Acta Math. 50, 189-358 (1927)]. In the early 1960s, he took advantage of the coincidence that several very talented students were working with him at Peking University at the same time to concentrate on what was by that time a rather neglected subject. The most significant of the papers they produced, in terms of the subsequent development of Nielsen theory, was written by *Boju Jiang* [Chin. Math. 5, 330-339 (1964; Zbl 0186.570)] which supplied an effective method for computing N(f) in many interesting settings. (The definition of the Nielsen number does not suggest how to compute it except in rather trivial cases.) With this tool, it became possible to construct the examples researchers would need to test their ideas.

By 1964, it seemed to Kiang that Nielsen fixed point theory had reached a stage of sufficient maturity to merit a book-length exposition. The history of this project, which reflects all too well the history of China in the 1960s and 70s, is recounted in a fascinating "Epilogue" to the present volume. Finally, with the help of some members of his group from the 1960s, Kiang finished the Chinese edition of this book in 1979. Subsequently, in spite of his advanced age and frail health, he set out to translate the book into English, to produce the volume presently under review.

The goal of this book is to introduce the student to Nielsen fixed point theory, assuming as little prerequisite knowledge a possible. A series of Appendices, on the fundamental group, covering spaces, and simplicial approximation techniques, are designed to bridge the gap between elementary topology and the material of the text.

Just as the universal covering space of the circle can serve as a model for the general theory of covering spaces (as, for instance, in the book of *M. Greenberg* and *J. Harper* [Algebraic topology: A first course (1981; Zbl 0498.55001]), so an analysis of the fixed point theory of self-maps of the circle leads one in a very natural way to the basic concepts of Nielsen fixed point theory. The connection is not just formal, but reflects the fact that much of Nielsen theory can be viewed as the application to fixed point questions of relationships between covering spaces and the fundamental group. Kiang's first chapter is devoted just to the fixed point theory of the circle and, like the rest of the book, it stresses the geometric content of the ideas, which are relatively easy to visualize in this case. The chapter ends with some historical remarks on the development of the parts of fixed point theory, associated with the names of Lefschetz and Nielsen, that depend on algebraic topology. This sets the stage for the rest of the book, in particular the next chapter, which presents the Nielsen number and establishes its basic properties, now in the general context of maps of finite polyhedra. The third chapter presents a careful exposition of the pioneering computational technique of Jiang mentioned above.

Much of the significance of Nielsen fixed point theory lies in the fact that not only is the Nielsen number N(f) a lower bound for the number of fixed points of all maps homotopic to f, but that on many polyhedra (including all manifolds of dimension at least 3) it is a sharp bound in the sense that there is a map g homotopic to f which has exactly N(f) fixed points. For instance, if $N(f) = 0$ then there is a map homotopic to f which is fixed point free. The Nielsen number vanishes if f is a map of a simply-connected polyhedron whose Lefschetz number is zero, so Nielsen theory produces a (homotopy) converse to the Lefschetz fixed point theorem in this setting. Chapter 4, the longest and most difficult of the book, presents a treatment of this topic based on *Gen-hua Shi* [Chin. Math. 8, 234-243 (1966; Zbl 0186.571)]. For the case of maps in general, a paper written subsequent to the 1979 Chinese edition [*Boju Jiang*, Am. J. Math. 102, 749-763 (1980; Zbl 0455.55001)], improved Shi's results significantly. However, in the course of the development of this chapter, there is an exposition, for the first time in English, of deep investigations of Shi on the minimum number of fixed points of maps in the homotopy class of the identity map, on any finite polyhedron, a subject that is not addressed in Jiang's paper. This chapter, which Kiang wrote with Shi's assistance, uses 19 figures to help the reader penetrate the delicate geometry behind the arguments.

The final chapter is divided into two independent parts. The first presents examples from *D. McCord* [Pac. J. Math. 66, 195-203 (1976; Zbl 0324.55003)] of selfmaps of manifolds with vanishing Lefschetz numbers for which the converse of the Lefschetz theorem does not hold because the Nielsen number is nonzero. The covering space techniques used remind the reader of some of the ideas from early in the book. The rest of the chapter concerns not the fixed point question $f(x) = x$ but rather the number of solutions to $f(x) = a$ for a given point a in the range of f, which may now be different from the domain polyhedron. Based on *R. Brooks* [Am. J. Math. 95, 720-728 (1973; Zbl 0319.55015)] this material shows the underlying ideas of Nielsen theory applied to a related problem to obtain simple and attractive results.

There are two other book-length expositions of Nielsen fixed point theory [the reviewer, The Lefschetz fixed point theorem (1971; Zbl 0216.196) and *Boju Jiang*, Lectures on Nielsen fixed point theory (Contemp. Math., 14, 1983; Zbl 0512.55003)]. Both assume more background from the reader than Kiang's book does, so the reader interested in either of them would do well to read Kiang first. Also, although there is some overlap among the three books, they complement each other to a considerable extent. The exposition of Shi's work in Kiang's book is clearer as well as more complete than in the reviewer's book and the topics of the final chapter of Kiang do not appear in either of the other books.

The all-important figures in this book are clear and well-drawn. But the typesetting does not respect mathematical niceties. For instance, the symbol $t \in I$ is printed with $t \in$ ending on line and I beginning the next. Although annoying, this does not really interfere with the reader's understanding. The price may, however, interfere with the readers ability to own the book.

Since this is a text intended for beginning graduate students, it is a pity there are no exercises. In spite of its leisurely pace, especially at the beginning, and the author's great care and patience in exposition, this is still a book that must be read with pencil in hand and this in itself will teach the student much. Nevertheless, some exercises that would encourage the reader to explore the concepts a bit further would have been very welcome.

The author has contributed much to fixed point theory in the past, through his research and through his influence on his students. In the future, this fine book, now that it is available in English, should become the standard introduction to fixed point theory for students everywhere.

Reviewer: R.F.Brown

MSC:

55-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to algebraic topology
55M20 Fixed points and coincidences in algebraic topology
54H25 Fixed-point and coincidence theorems (topological aspects)

Keywords:

Nielsen number; covering spaces; fundamental group; Lefschetz number