
A vector field for a deformation of a Riemann surface or of a Fuchsian group may be supported only on a subset, $V$. The $\bar{\partial}$ derivative of the vector field is a Beltrami differential whose support will also be contained in the subset $V$. One may ask which local deformations in Teichmüller space can be obtained by deformations supported on $V$, assuming that the set $V$ is at least a set of positive measure. This article investigates two problems:

(A) Under what conditions on $V$ do the local deformations supported on $V$ cover a neighborhood of the origin in Teichmüller space? and

(B) Under what conditions on $V$ is the origin in Teichmüller space not an interior point of the deformation supported on $V$?

The answer to (A) is that $V$ must be large and to (B) is that $V$ must be small. In an elegant and natural way, the author makes these answers precise by using the Poincaré metric and the Bergman kernel. For the sake of simplicity, consider only the case of a Riemann surface $\Omega$ and assume $V$ is contained in $\Omega$. Let $F(\zeta, z)\begin{pmatrix} d\zeta^2 \\ d\bar{\zeta}^2 \end{pmatrix}$ be the Bergman kernel, $\lambda(z)|dz|$ be the Poincaré metric and $\omega(z)$ be the density function defined by

$$\omega(z) = \lambda(z)^{-2} \sup_{\zeta} \lambda(\zeta)^{-2} |F(\zeta, z)|.$$

The author’s answer to question (A) is that if $V$ is large in the sense that

$$\int\int_{\Omega-V} \max\{\omega(z)^2, 1\} \lambda(z)^2 dxdy < \infty,$$

then the deformations supported on $V$ and bounded by an arbitrary $\delta > 0$ cover a neighborhood of the origin. There is a further variational interpretation of this result. A useful corollary follows from showing that (1) is fulfilled if the non-Euclidean area of $\Omega-E$ is finite and if the injectivity radius of $\Omega$ is positive. Another corollary follows from showing that (1) is fulfilled if $\Omega-V$ is relatively compact in the Riemann surface obtained by adding the punctures of $\Omega$ to it.

The answer to (B) is interesting only when the Teichmüller space is infinite dimensional, since, if $V$ has positive measure, it is easy to show that when the Teichmüller space is finite dimensional the deformations supported in $V$ contain a neighborhood of the identity. For a Beltrami coefficient $\mu$ with $\|\mu\|_{\infty} = 1$, let $\Delta(\mu)$ be the set of all points in the Teichmüller space obtainable by a deformation $t\mu$ where $t$ is a complex number of absolute value less than one. The author shows that if $V$ is a small set in the sense that

$$\int\int_{V} \omega(z)\lambda(z)^2 dxdy < \infty,$$

then for any $\mu$ which possesses a degenerating Hamilton sequence the set $\Delta(\mu)\backslash\{0\}$ is disjoint from all of the deformations realizable by Beltrami coefficients supported in $V$. The article contains numerous further applications and generalizations of these results.

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MSC:

- 30F35 Fuchsian groups and automorphic functions (aspects of compact Riemann surfaces and uniformization)
- 32G15 Moduli of Riemann surfaces, Teichmüller theory (complex-analytic aspects in several variables)
- 30C62 Quasiconformal mappings in the complex plane

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