Drezet, J.-M.; **Narasimhan, M. S.**
Groupe de Picard des variétés de modules de fibrés semi-stable sur les courbes algébriques. (Picard groups of moduli varieties of semi- stable bundles on algebraic curves). (French)
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Let X be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Let U(r,d) (resp. $U_s(r,d)$) be the moduli space of algebraic semistable vector bundles (resp. the open subset corresponding to the stable bundles) of rank $r \geq 2$ and degree d over X. It is known that $U(r,d)$ is a normal, irreducible, projective variety. If $gcd(g,r) \neq 1$ and one excludes also the case $g = r = 2$, d even then $U(r,d)$ is not smooth, $Sing(U(r,d)) = U(r,d) \setminus U_s(r,d)$ and $co\dim_{U(r,d)} U(r,d) \setminus U_s(r,d) \geq 2$. For $L \in Pic(X)$, $\deg(L) = d$ let denote by U(r,L) (resp. $U_s(r,L)$) the closed subvariety of $U(r,d)$ (resp. $U_s(r,d)$) corresponding to the vector bundles with determinant isomorphic to L. The aim of this paper is to give a complete description of $Pic(U(r,d))$ and $Pic(U(r,L))$ when $gcd(g,r) \neq 1$ and $(g,r) \neq (2,2)$, d even.

The first result is that even they are singular, $U(r,d)$ and $U(r,L)$ are locally factorial. Let now $\gcd(r,d) = n$ and let $\mathcal{F}$ be a vector bundle on X such that $\deg(\mathcal{F}) = (-d + r(g-1))/n$ and $rk(\mathcal{F}) = r/n$. Then $\chi(\mathcal{E} \otimes \mathcal{F}) = 0$ for all vector bundles $\mathcal{E}$ on X of rank r and degree d. One can show that $\mathcal{F}$ above can be chosen such that there exists $\mathcal{E} \in U_s(r,d)$ with $H^0(X, \mathcal{E} \otimes \mathcal{F}) = H^1(X, \mathcal{E} \otimes \mathcal{F}) = 0$. Then for such an $\mathcal{F}$ denote by $\Theta^s_{\mathcal{F}}$ (respectively $\Theta^s_{\mathcal{F},L}$) the set of points of $U_s(r,d)$ (resp. $U_s(r,L)$) which correspond to stable bundles $\mathcal{E}$ with $H^0(X, \mathcal{E} \otimes \mathcal{F}) \neq 0$. These are showed to be hypersurfaces in $U_s(r,d)$ respectively in $U_s(r,L)$. Their closure in $U(r,d)$ (respectively $U(r,L)$) are denoted by $\Theta_{\mathcal{F}}$ (resp. $\Theta_{\mathcal{F},L}$) and called theta divisors.

The line bundle $\mathcal{O}(\Theta_{\mathcal{F},L})$ is independent of the choice of $\mathcal{F}$ and $Pic(U(r,L))$ is isomorphic to $\mathbb{Z}$ having $\mathcal{O}(\Theta_{\mathcal{F},L})$ as generator. Let $I^{(d)}$ be the Jacobian of the line bundles of degree d on X. Then, through the canonical morphism $\det : U(r,d) \to I^{(d)}$, $Pic(I^{(d)})$ is seen as a subgroup of Pic(U(r,d)) and one has the isomorphism $Pic(U(r,d)) \cong Pic(I^{(d)}) \oplus \mathbb{Z}\mathcal{O}(\Theta_{\mathcal{F}})$. Here $\mathcal{O}(\Theta_{\mathcal{F}})$ is dependent on the choice of $\mathcal{F}$: $\mathcal{O}(\Theta_{\mathcal{F}'}) \cong \mathcal{O}(\Theta_{\mathcal{F}}) \otimes \det^*(\det \mathcal{F}' \otimes (\det \mathcal{F})^{-1})$.

The paper also contains a complete description of the dualizing sheaves of $U(r,L)$ and $U(r,d)$ and a proof of the nonexistence of Poincaré bundles on open subsets of the moduli space $M_s(\mathbb{P}_2(\mathbb{C}), r, c_1, c_2)$ in case r, c_1 and χ are not prime to each other.

Reviewer: Sorin Popescu

MSC:

14C22 Picard groups
14H10 Families, moduli of curves (algebraic)
14F05 Sheaves, derived categories of sheaves, etc. (MSC2010)

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