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The growth of $A$-subharmonic functions and quasiregular mappings along asymptotic paths.
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A function $u$ in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said to be $A$-harmonic if it is a continuous weak solution to the quasilinear degenerate elliptic equation (*) $\text{div}(A(x,\nabla u)) = 0$ in $\Omega$. Here $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping with $A(x,h) \cdot h \approx |h|^p$ for some $p \in (1,\infty)$; the precise assumptions on $A$ are given in the paper.

A function $u$ in $\Omega$ is said to be $A$-subharmonic if (i) $u$ is upper semicontinuous; (ii) $-\infty \leq u < \infty$; and (iii) for each domain $D$ compactly contained in $\Omega$ and each $A$-harmonic $h \in C(\bar{D})$, $u \leq h$ in $\partial D$ implies $u \leq h$ in $D$.

$A$-subharmonic functions and their (nonlinear) potential theory were studied recently by Heinonen and Kilpeläinen. The conformally invariant borderline case $p = n$ was examined earlier by Granlund, Lindqvist and Martio.

Equation (*) is modeled by the $p$-Laplace equation (***) $\text{div}(\nabla u) = 0$ in which case the terminology $p$-harmonic or $p$-subharmonic is customary. When $p = 2$, this equation reduces to the familiar Laplace equation.

The nonlinear potential theory based on $A$-subharmonic functions is an extension of the classical theory of subharmonic functions in $\mathbb{R}^n$. Many results from the linear theory have found their natural generalizations in the nonlinear setting although new methods of proof are often unavoidable.

In this paper, we study the growth of entire $A$-subharmonic functions along asymptotic paths. It was shown by Heinonen that if $u$ is $A$-subharmonic in $\mathbb{R}^n$ and unbounded from above, there exists an asymptotic path for $u$. That is, there exists a path tending to infinity along which $u$ tends to infinity. For ordinary subharmonic functions, i.e., $A(x,\nabla u) = \nabla u$, this problem has of late been studied rather extensively and our results generalize works of K. A. Barth, D. A. Brannan, W. K. Hayman, A. E. Eremenko and others.

Thus, let $u$ be an entire $A$-subharmonic function in $\mathbb{R}^n$, unbounded from above. We make the assumption that the set $\{u < 0\}$ is sufficiently thick at infinity, the thickness measured either in terms of the $p$-capacity or in terms of the angle measure on spheres. Our main results state that there is an asymptotic path on which $u(x) \geq |x|^\gamma$ for some $\gamma > 0$. It will be shown furthermore that $\gamma$ tends to infinity if the set $\{u \geq 0\}$ narrows appropriately at infinity.

For the usual Laplacian, $p = 2$ in (**), the imposed capacity density condition is nearly sharp. Even for harmonic functions in space, results of Ancona, Hayman and Wu shows that no minimum growth along paths can be expected in general.

The potential theory of $A$-subharmonic functions and the theory of quasiregular mappings in $\mathbb{R}^n$ are naturally linked. For instance if $f$ is a quasiregular mapping, then akin to the classical plane case, $\log|f|$ is $A$-subharmonic for an appropriate $A$ with $A(x,h) \cdot h \approx |h|^n$. Especially, if $f$ is an entire K-quasiregular mapping which possesses at least one finite asymptotic value, then it follows from our results that there is an asymptotic path on which $\log|f(x)| \geq |x|^\gamma$ where $\gamma = \gamma(n,K) > 0$. This improves a result of S. Rickman and M. Vuorinen [Ann. Acad. Sci. Fenn., Ser. A I 17, 221-231 (1982; Zbl 0516.30018)] where they proved that such an $f$ must have a positive lower order.

Reviewer: J. Rossi

MSC:
31B99 Higher-dimensional potential theory
35J70 Degenerate elliptic equations
31B15 Potentials and capacities, extremal length and related notions in higher dimensions
31B05 Harmonic, subharmonic, superharmonic functions in higher dimensions
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