Furstenberg, Hillel

Nonconventional ergodic averages. (English) Zbl 0711.28006

The legacy of John von Neumann, Proc. Summer Res. Inst., Hempstead/NY (USA) 1988, Proc. Symp. Pure Math. 50, 43-56 (1990).

[For the entire collection see Zbl 0699.00010.]

Let T be a measure preserving transformation defined on the measure space (X,$\mathcal{B}$, μ). If T is invertible the operator U_T, $U_T f(x) = f(Tx)$, defines a unitary operator on the Hilbert space $L^2(X, \mathcal{B}, \mu)$. This observation due to B. O. Koopman led John von Neumann to his proof of the mean ergodic theorem concerning ergodic averages

$$(1/N)(f + U_T f + U_T^2 f + ... + U_T^{N-1} f).$$

The paper under review discusses a particular direction in which such ergodic theorems have developed in recent years. Specifically the author studies nonconventional ergodic averages of the following types.

$$(a)\quad (1/N)\sum_{n=0}^{N-1} U_T^{n^2} f, \quad (b)\quad (1/N)\sum_{n=0}^{N-1} U_T^n f U_T^{2n} g,$$

$$(c)\quad (1/N)\sum_{n=0}^{N-1} U_T^n f U_T^{2n} g U_T^{3n} h, \quad (d)\quad (1/N)\sum_{n=0}^{N-1} U_T^n f U_T^{n^2} g.$$

The limits as $N \to \infty$ for (a) and (b) are shown to exist (for approximately bounded functions) in $L^2(X, \mathcal{B}, \mu)$. A discussion of (c) (whose limit is known to exist in $L^2(X)$) is also given, but the limit (d) has not yet been established.

It is pointed out that these limits are governed by factors similar to Kronecker factors, but unlike the case of the mean ergodic theorem, they need not be determined by the spectrum of U_T. Furthermore the result of (b) established a new proof of Roth's theorem: If $S \subseteq \mathbb{Z}$ has positive upper density then it must contain a three element arithmetic progression. Another result of this type mentioned is that such sets S always contain solutions to the equation $x - y = (y - z)^2$.

Details of the limit (c) and related results will appear in a joint work of the author and B. Weiss. Other related results appear in work by *J.-P. Conze* and *E. Lesigne* [C. R. Acad. Sci., Paris, Sér. I 306, No.12, 491-493 (1988; Zbl 0641.28010) and Ergodic Theory Dyn. Syst. 9, 115-126 (1989)]. *J. Bourgain* has shown that averages of the form (a) and (b) converge a.e. (to appear). Finally we should mention that the study of nonconventional ergodic averages was initiated by the author, and an introduction to these ideas was given in his book "Recurrence in ergodic theory and combinatorial number theory" (1981; Zbl 0459.28023)].

Reviewer: G.R.Goodson

MSC:

28D05 Measure-preserving transformations
11B25 Arithmetic progressions

Keywords:

mean ergodic theorem; ergodic averages; Roth's theorem; arithmetic progression; nonconventional ergodic averages