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Geometrization of 3-dimensional orbifolds. (English) Zbl 1087.57009

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In 1982 Thurston announced the orbifold theorem or geometrization theorem for 3-orbifolds with non-empty singular set (ramification locus). After various general outlines of the proof, notes on the proof and proofs of special cases (most notably the solution of the classical Smith conjecture), the present paper offers the first complete published proof of this important theorem (for a different approach, see [*D. Cooper, C. D. Hodgson* and *S. P. Kerckhoff*, Three-dimensional orbifolds and cone-manifolds. Math. Soc. Japan Mem. 5, Tokyo (MSJ) (2000; Zbl 0955.57014)]. Since a 3-orbifold with empty singular set is a 3-manifold, the orbifold theorem complements and extends the 3-manifold geometrization theorem, with major steps due to Thurston for Haken 3-manifolds and, recently, Perelman for the general case.

A 3-orbifold is *geometric* if either its interior has one of Thurston's eight 3-dimensional geometries (constant curvature, Seifert fibered or solvable geometry SOL), or it is the quotient of the 3-ball by a finite orthogonal action. The orbifold theorem then states that every compact, orientable, irreducible, *topologically atoroidal* 3-orbifold with non-empty singular set is geometric (a 3-orbifold is topologically atoroidal if it does not contain an embedded essential orientable toric (Euclidean) 2-suborbifold). By known decomposition theorems, any compact orientable 3-orbifold without bad 2-suborbifolds can be decomposed along spherical and Euclidean 2-suborbifolds into irreducible atoroidal 3-orbifolds which, by the orbifold theorem, are geometric if the singular set is non-empty. As a consequence, any orientation-preserving smooth non-free finite group action on S^3 is conjugate to an orthogonal action (generalizing the classical Smith conjecture regarding the case of cyclic groups; the case of free actions on S^3 is a major case in Perelman's work); also, every compact orientable 3-orbifold without bad 2-suborbifolds is the quotient of a compact 3-manifold by a finite group action.

The main theorem proved in the present paper (which implies the orbifold theorem) is the uniformization or geometrization of *small* 3-orbifolds: a compact orientable 3-orbifold is small if its boundary (perhaps empty) consists of turnovers (2-spheres with three branching points), and it does not contain any other closed incompressible orientable 2-suborbifold. Every compact orientable irreducible and atoroidal 3-orbifold can be canonically split along a maximal collection of hyperbolic turnovers, and the resulting pieces are either Haken or small 3-orbifolds. Using also an extension of Thurston's hyperbolization theorem to the case of Haken 3-orbifolds, the uniformization theorem for small 3-orbifolds then implies the orbifold theorem.

The proof of this main theorem (the uniformization theorem for small 3-orbifolds) is reduced to the case when the complement of the singular part of the orbifold is hyperbolic. This hyperbolic structure on the complement is viewed as having cone angles zero around the singular edges. These cone angles are then increased by deforming the hyperbolic structure. If a resulting sequence of such hyperbolic cone-manifolds does not collapse (the injectivity radii for the cone structures do not converge to zero) then it is shown that the original orbifold cone angles can be reached in the deformation space of hyperbolic cone structures, and hence the orbifold is hyperbolic. So there remains the case when the sequence of cone manifolds collapses. Then, if the diameters of the collapsing cone structures are bounded away from zero, a fibration theorem shows that the orbifold is Seifert fibered. Finally, if the diameters of the sequence of cone structures converge to zero, it is shown that the orbifold is geometric; there occurs the case here that the orbifold is closed and admits a euclidean cone structure with cone angles strictly less then its orbifold angles; in this case, using Hamilton's result about the Ricci flow on 3-manifolds, it is shown that the orbifold is spherical.

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