Morel, Fabien

The stable $\mathbb{A}^1$-connectivity theorems. (English) Zbl 1117.14023

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A sheaf of S^1-spectra E on the category of smooth schemes over S in the Nisnevich topology is said to be (-1)-connected if all its negative homotopy sheaves vanish. One says that the stable $\mathbb{A}^1$-connectivity property holds over S if the $\mathbb{A}^1$-localization functor preserves (-1)-connected sheaves. The main result of the paper is that the stable $\mathbb{A}^1$-connectivity property holds when the base S is the spectrum of a field. The proof essentially uses Gabber's presentation lemma over infinite fields. As a consequence the author proves that for any sheaf of spectra E defined over a field its $\mathbb{A}^1$-homotopy sheaves are strictly $\mathbb{A}^1$-invariant. In particular, it holds for the sheaf of Balmer-Witt groups. In the language of stable homotopy categories it also implies that there is a t-structure on the stable $\mathbb{A}^1$-homotopy category of S^1-spectra whose heart consists of strictly $\mathbb{A}^1$-invariant sheaves. This t-structure can be viewed as a direct analogue in the stable $\mathbb{A}^1$-homotopy theory of Voevodsky's homotopy t-structure for the triangulated category $\mathrm{DM}^{\mathrm{eff}}$ over a perfect field. As an important application the author proves the Gersten conjecture for pure sheaves over a field, e.g., strictly $\mathbb{A}^1$-homotopy invariant sheaves. He also discusses the finitness properties of $\mathbb{A}^1$-homotopy groups.

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References:

[1] Beilinson A., Bernstein. and Deligne P. Faisceaux pervers, (French) [Perverse sheaves] Analysis and Topology on Singular Spaces, I (Luminy, 1981), Astérisque, Vol. 100, Soc. Math. France, Paris (1982).

[2] Bousfield A.K., Friedlander E.M.: Homotopy theory of {\(\Gamma\)}-spaces, spectra, and bisimplicial sets, in Geometric Applications of Homotopy Theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., Vol. 658, Springer, Berlin (1978), pp. 80–130 · Zbl 0405.55021

[3] Brown K.S., Gersten S.M.: Algebraic K-theory as Generalized Sheaf Cohomology. in: Algebraic K-Theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972) Lecture Notes in Math., Vol. 341, Springer, Berlin (1973), pp. 266–292

[4] Bloch S., Ogus A. Gersten's conjecture and the homology of schemes, Ann. Sci. Ec. Nom. Sup. 4 série 7, 181–202 · Zbl 0307.14008

[5] Colliot-Thélène J.-L., Hoobler R.T., Kahn B. The Bloch–Ogus–Gabber theorem. (English. English summary) Algebraic K-theory (Toronto, ON, 1996), Fields Inst. Commun., Vol. 16, Amer. Math. Soc., Providence, RI (1997), pp. 31–94. · Zbl 0911.14004

[6] Déglise, F.: Modules homotopiques avec transferts et motifs génériques, Thèse de l'Université Paris 7, 2002 Available at http://www.math.jussieu.fr/\(\sim\)deglise/.

[7] Déglise, F.: Modules de cycles et motifs mixtes, C.R. Acad. Sci. Paris, t. Comptes rendus Mathématiques, t. 336, (Série I), 41–46. · Zbl 1042.19001

[8] Faltings G. Finiteness Theorems for Abelian Varieties over Number Fields, in G. Cornell and J. Silverman (eds.), Arithmetic Geometry Springer–Verlag (1986). · Zbl 0602.14044

[9] Grothendieck A., Dieudonné, J.: Éléments de géométrie algébrique. IV, \(\backslash\)'Etude locale des schémas et des morphismes de schémas IV, (French) Inst. Hautes Études Sci. Publ. Math. No. 32 (1967), 361 pp.

[10] Goerss P., Jardine J.F.: Simplicial Homotopy Theory, Progress in Mathematics, 174. Birkhäuser Verlag, Basel (1999), xvi+510 pp. · Zbl 0949.55001

[11] Hornbostel J. A1-representability of hermitian K-theory and Witt groups, Preprint, available at http://www.math.uiuc.edu/K-theory/0578/ . · Zbl 1078.19004

[12] Hu P. S-modules in the category of schemes, Mem. Am. Math. Soc. 161 (767) (2003)

[13] Kato K. Milnor K-theory and the Chow group of zero cycles, in Applications of Algebraic K-theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colo., 1983), 241–253, Contemp. Math., 55, Amer. Math. Soc., Providence, RI (1986).

[14] Knus M.-A.: Quadratic and Hermitian forms over Rings, With a foreword by I. Bertuccioni. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 294 Springer-Verlag, Berlin (1991).

[15] Jardine J.F.: Motivic symmetric spectra, (English. English summary) Doc. Math. 5 (2000), 445–553 (electronic). · Zbl 0969.19004

[16] May J.P.: Simplicial Objects in Algebraic Topology. Reprint of the 1967 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL (1992). · Zbl 0769.55001

[17] Morel F. Théorie homotopique des schémas, Astérisque (1999).

[18] Morel F. Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques C.R. Acad. Sci. Paris, t. 328 (Série I), (1999), 963–968 · Zbl 0937.19002

[19] Morel F. An introduction to \(\mathbb{A}^1\) -homotopy theory, in Karoubi, Kuku and Pedrini (eds), Contemporary Developments in Algebraic K-theory, ICTP Lecture notes, Vol. 15 (2003). · Zbl 1081.14029

[20] Morel F. On the motivic {\(\pi\)}0 of the sphere spectrum, in J.P.C. Greenlees (ed.), Axiomatic, Enriched and Motivic Homotopy Theory, Kluwer Academic Publishers (2004), 219–260 · Zbl 1130.14019

[21] Morel F. On the structure of \(\mathbb{A}^1\) -homotopy sheaves I \& II, preprints, available at http://www.mathematik.uni-muenchen.de/\(\sim\)morel/listepublications.html

[22] Morel F. Rationalized motivic sphere spectrum and rational motivic cohomology, in preparation.

[23] Morel F., Voevodsky V. : \(\mathbb{A}^1\) -homotopy theory of schemes, Publications Mathématiques de l'I.H.É.S. no 90.

[24] Nisnevich Y. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory, in Algebraic K-Theory: Connections with Geometry and Topology (Lake Louise, AB, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 279, Kluwer Acad. Publ., Dordrecht (1989), pp. 241–342

[25] Quillen D.G.: Higher algebraic K-theory. I, in Algebraic K-theory, I: Higher K -theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., Vol. 341, Springer, Berlin (1973), pp. 85–147 · Zbl 0292.18004

[26] Suslin A., Voevodsky V. Bloch–Kato conjecture and motivic cohomology with finite coefficients, The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., Vol. 548, Kluwer Acad. Publ., Dordrecht (2000), pp. 117–189 · Zbl 1005.19001

[27] Suslin A., Voevodsky V. Relative cycles and Chow sheaves, in Cycles, transfers, and Motivic Homology Theories, Ann. of Math. Stud., Vol. 143, Princeton Univ. Press, Princeton, NJ (2000), pp. 10–86. · Zbl 1019.14004

[28] Voevodsky V. Cohomological theory of presheaves with transfers, in Cycles, Transfers, and Motivic Homology Theories, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ (2000), pp. 87–137 · Zbl 1019.14010

[29] Voevodsky V. Triangulated categories of motives over a field, in Cycles, Transfers, and Motivic Homology Theories, Ann. of Math. Stud., Vol. 143, Princeton Univ. Press, Princeton, NJ (2000), pp. 188–238 · Zbl 1019.14009

[30] Voevodsky V. The \(\mathbb{A}^1\) -homotopy theory, Proceedings of the international congress of mathematicians, Berlin (1998). · Zbl 0907.19002

[31] Voevodsky V. Cancellation theorem, preprint, available at http://www.math.uiuc.edu/ K-theory/0541/index.html . · Zbl 1202.14022

[32] Voevodsky V. Unstable motivic homotopy categories in Nisnevich and cdh-topologies, Preprint, available at http://www.math.uiuc.edu/K-theory/0444/ . · Zbl 1187.14025