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Riesz transform on manifolds and Poincaré inequalities. (English) Zbl 1116.58023

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The authors study the validity of the L^p inequality for the Riesz transform when $p > 2$ and of its reverse inequality when $1 < p < 2$ on complete Riemannian manifolds under the doubling property and some Poincaré inequality.

Let M be a non-compact complete Riemannian manifold. Consider the L^p inequality

$$|| \, |\nabla \Delta^{-1/2} f| \, ||_p \leq C_p ||f||_p, \tag{1}$$

and the "reverse inequality"

$$||\Delta^{1/2} f||_p \leq C_p || \, |\nabla f| \, ||_p. \tag{2}$$

A manifold is said to satisfy the *volume doubling property* if there exists a constant C such that for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

It satisfies the *scaled Poincaré inequalities* P_p if there exists $C > 0$ such that for every ball $B(x, r)$, and every f with f, ∇f locally p-integrable, then

$$\int_B |f - f_B|^p d\mu \leq Cr^p \int_B |\nabla f|^p \, d\mu, \tag{P_p}$$

where f_B is the mean of f on B.

The main theorem is: Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property, and (P_2). Then there exists $\epsilon > 0$ such that (1) holds for $2 < p < 2 + \epsilon$.

An important step in the proof is the following theorem: Let M be a complete non-compact Riemannian manifold satisfying the volume doubling property and P_q for some $q \in [1, 2]$. Then (2) holds for $q < p < 2$. If $q = 1$, there is a weak-type $(1, 1)$ estimate.

Reviewer: Christine Guenther (Forest Grove)

MSC:

58J35	Heat and other parabolic equation methods for PDEs on manifolds	Cited in **2** Reviews
42B20	Singular and oscillatory integrals (Calderón-Zygmund, etc.)	Cited in **34** Documents

Full Text: EuDML

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