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**Conformal and potential analysis in Hele-Shaw cells.** (English) Zbl 1122.76002

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The general theme of the book is mathematical treatment of unsteady Hele-Shaw flows with methods of geometric function theory and two-dimensional potential theory. It is a research monograph and good survey of recent developments of the subject. Many parts of the book reflects areas of research activities of the authors themselves during the past two to three decades.

The classical Hele-Shaw flow model, as usually described in standard textbooks on fluid mechanics, comprises steady flow of viscous incompressible Newtonian fluid in a narrow gap between two parallel plates in which a cylindrical body of arbitrary cross-sectional shape may be spanned perpendicular to the plates. Such an arrangement, in which the plates (at least one of them) are usually made of transparent material for purposes of visualisation, now bears the name Hele-Shaw cell in honor of its originator. In 1898, namely, Hele-Shaw first experimentally demonstrated that, under the conditions of unidirectional far-field flow parallel to the plates and sufficiently small Reynolds number based on the gap width (Stoksian flow), the streamlines of the flow around the cylindrical obstacle in Hele-Shaw cell appear the same as in a hypothetical plane potential flow of an inviscid fluid around a body of the same form. This is a remarkable relation between a flow with dominant viscous forces and the inviscid zero-circulation potential flow around a closed planar body contour. "A mathematical proof of the identity of the streamlines obtained by means of a viscous film with those of a perfect fluid moving in two dimensions" was given by G. Stokes in an appendix to the Hele-Shaw's short description (1898) of his experimental device.

The mathematical formulation of horizontal Hele-Shaw flow averaged across the gap height is given by the equation $\mathbf{v} = -\mathrm{const} \cdot \nabla p$ which is now often referred to as Hele-Shaw equation and in which in turn one recognizes the analogy with Darcy's (1856) phenomenological law governing flows through porous media. The peculiarity of the Hele-Shaw equation is that the velocity potential is just the pressure which, corresponding with its physical nature, must be a single-valued and everywhere continuous function, whereas the potential function of inviscid flows otherwise is allowed to be discontinuous on interfaces and is even multi-valued in flows with circulation.

Besides being a model for studying potential flow patterns around variously shaped bodies, there have been many other areas of application of (steady or unsteady) Hele-Shaw analog (also named viscous flow analog or parallel-plates analogy). Especially interesting from mathematical point of view are applications in which the real process may be with good approximation assumed to be governed by an analogous unsteady Hele-Shaw flow model for which the Hele-Shaw equation is the same as above, with the only difference that velocity and pressure are now additionally dependent on time. This can only be valid when the local acceleration force in the analogous real fluid flow may be neglected compared to the viscous force, i.e. when the real flow develops in time very slowly.

The unsteady Hele-Shaw problems on which this book is generally focusing are moving free-boundary problems governed by unsteady Hele-Shaw equation, of which the "simplest" fluid dynamical case that was first studied by Polubarinova-Kochina and Galin as an analog to problems connected with ground water spreading and oil production from reservoirs. The situation can be briefly described as follows. At the initial time a viscous liquid occupies a bounded simply connected plane domain surrounded by inviscid constant-pressure fluid. The flow is effected by a point source or sink located in the viscous fluid domain. One would like to know how such a flow domain evolves with the time. In this problem, at every time instant the pressure function must satisfy Laplace equation in the punctured flow domain and the constant-value condition on the moving boundary (Dirichlet problem). Additionally, the kinematic boundary condition, which in this case is actually Hele-Shaw equation multiplied by unit outer normal vector to the boundary, must also be satisfied. The indirect complex variable method as proposed by Polubarinova-Kochina and Galin in the mid 1940s (Polubarinova-Galin equation) for solving this zero surface tension Hele-Shaw problem (also named Laplacian growth model) takes a prominent place in the book.

The book is divided into seven chapters. The first chapter (Introduction and background) provides the necessary basics of fluid mechanics and conformal mapping. It also contains derivation of Hele-Shaw and Polubarina-Galin (P-G) equations as well as discussion of existence, ill/well-posedness, singularity formation, regularization, and numerical treatment of the problem. The Löwner-Kufarev type equation, Richardson's complex moments, Schwarz function, and a general solution via variable substitution are also considered in this chapter in view of their connection with the Polubarina-Galin equation.

In the second chapter (Explicit solutions) several strong solutions to Polubarinova-Galin equation are presented: the Polubarinova-Galin cardioid, Saffman-Taylor's fingers, solutions produced by using rational mapping functions, corner flows with Robin's boundary conditions. The definition of strong solutions is elaborated, and the existence and uniqueness theorems proved for rational solutions.

In the third chapter (Weak solutions and balayage) weak solutions are defined and their properties vis-a-vis strong solutions are elaborated. The existence and uniqueness theorems for weak solutions are stated and proved. The close connection between potential theory (partial balayage, quadrature domain theory) and Hele-Shaw flows is elaborated and used throughout this and later chapters. Another section of this chapter is devoted to existence and uniqueness of weak and strong solutions backward in time.

In the next chapter (Geometric properties) the authors investigate several questions regarding general Hele-Shaw flows, like: Which is the minimal distance from the source to the evolving free boundary, or from points in the initial domain to points on the evolving boundary? Which classes of univalent mapping functions that satisfy the P-G equation produce starlike or convex phase domain? Which geometric properties are preserved during the time evolution of the moving boundary? What are the geometric features of weak solutions? The Hele-Shaw problem with surface tension and the problem of solidification/melting in forced potential flow are considered in the context of this chapter, too.

The chapters (Capacities and isoperimetric inequalities) contains results on the relations between some characteristics of the moving boundary and the rate of area change in several models of Hele-Shaw flows: Hele-Shaw cell with obstacles (Robin's capacity and Robin's reduced modulus), corner flow (derivation of an isoperimetric inequality), melting of a bounded crystal.

The sixth chapter (General evolution equations) contains a collection of results on general subordination dynamics described through the Löwner-Kufarev equation. The afore-mentioned Löwner-Kufarev type equation for Hele-Shaw flows describes monotone evolution of simply connected phase domains as conformal univalent maps from the unit circle, and is only a special case of the classical Löwner-Kufarev equation. Here the authors study evolution equations for $t$-parametric conformal maps with quasiconformal extensions, in particular, maps smoothly extendible onto the unit circle. The approach is based on the study of flows on the universal Teichmüler space and on the Kählerian manifold embedded into it. The diffusion-limited aggregation model and fractal growth are also discussed in the final section of this chapter.

In the last chapter (Hele-Shaw evolution and strings) the authors give an introduction to the classical string theory and show that it can be applied to the Hele-Shaw subordination evolution.

Up-to-date reference is made to the literature through the enclosed bibliography of 345 items. The reader is also referred to the 600-entry bibliography for free and moving boundary problems for Hele-Shaw and Stokes flows covering the period 1898–1998, collected and electronically published be Gillow and Howison (Ref. [125] in the book). The book closes with a list of symbols and a keyword index.

In the preface, the authors state that for most parts of the book they assume the background provided by graduate courses in real and complex analysis, in particular, the theory of conformal mappings and in fluid mechanics. However, it must be added that large part of the book contains rather advanced material and is less easy to read.

The authors have made a major effort to keep the book as self-contained as possible. In general, the authors' presentation of principal ideas, methods and results is at varying degrees of depth and broadness, with ample references to original works for more detailed and complementary information.

Unfortunately, the book is not free of typographical errors, lapses or oversights, as seemingly no books completely are. It is rather annoying that at numerous places in the book it incorrectly stands $z$ instead of $\zeta$. Then there are references to a non-existing theorem 5.1.2, use of $M$ and $\mathcal{M}$ for the same Kählerian manifold, typos in the running title of chapter 6 and in reference [39], etc. It is also to be mentioned that the authors do not explain why in writing this book they decided, contrary to the usual praxis, to take the strength of a sink positive and of a source negative.

Another noteworthy feature of this book are the informative, occasionally piquant, biographical and historic-contextual footnotes about authors who made contributions to a specific topic (Did we know the political orientation of Bieberbach and Teichmüller?). Also welcome are the etymological explanations of some notions and where to read about them (Where does the word balayage come from? Who was Robin?).

In summary, this is a welcome book to be recommended to a wide group of readers, from graduate students to applied scientists and mathematicians.

Reviewer: Tomislav Zlatanovski (Skopje)

**MSC:**

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| 76-02 | Research exposition (monographs, survey articles) pertaining to fluid mechanics |
| 76D27 | Other free boundary flows; Hele-Shaw flows |
| 76M40 | Complex variables methods applied to problems in fluid mechanics |
| 30C20 | Conformal mappings of special domains |
| 30C62 | Quasiconformal mappings in the complex plane |
| 31A05 | Harmonic, subharmonic, superharmonic functions in two dimensions |



**Keywords:**

Polubarinova-Galin equation; Laplacian growth model; Löwner-Kufarev equation; balayage